QUADRUPLES, ADMISSIBLE ELEMENTS AND HERRMANN'S ENDOMORPHISMS

RAFAEL STEKOLSHCHIK

Abstract. The notion of a *perfect element* of the free modular lattice D^r generated by $r \geq 1$ elements is introduced by Gelfand and Ponomarev in [GP74]. The classification of such elements in the lattice D^4 (and also in D^r, where $r > 4$) is given in [GP74], [GP76], [GP77].

Admissible elements are the building blocks used for the construction of perfect elements. In [GP74], Gelfand and Ponomarev construct admissible elements of the lattice D^r recurrently. We suggest a direct method for creating admissible elements in D^4 and show that these elements coincide modulo linear equivalence with admissible elements constructed by Gelfand and Ponomarev. Admissible sequences and admissible elements for D^4 form 8 classes and possess some periodicity.

The main aim of the present paper is to find a connection between admissible elements in D^4 and *Herrmann's endomorphisms* γ_{1i}. Herrmann in [H82] constructed perfect elements s_n, t_n, $p_{i,n}$ in D^4 by means of some *endomorphisms* γ_{ij} and showed that these perfect elements coincide with the Gelfand-Ponomarev perfect elements modulo linear equivalence. We show that the admissible elements in D^4 are also obtained by means of Herrmann's endomorphisms γ_{ij}. Herrmann's endomorphism γ_{ij} and the *elementary map* of Gelfand-Ponomarev ϕ_i act, in a sense, in opposite directions, namely the endomorphism γ_{ij} adds the index to the start of the admissible sequence, and the elementary map ϕ_i adds the index to the end of the admissible sequence, see (1.2) and (1.3).

Contents

1991 *Mathematics Subject Classification.* 16G20, 06C05, 06B15.
Key words and phrases. Modular lattices, Perfect polynomials, Quadruples.
email: rs2@biu.013.net.il.

List of Figures

List of Tables

1. **Introduction**

We denote by D^4 the free modular lattice with four generators. Indecomposable representations of D^4 were classified by Nazarova in [Naz67] and Gelfand-Ponomarev in [GP70][1].

In [GP74] Gelfand-Ponomarev constructed *perfect elements* in D^4. We recall that an element $a \in D$ of a modular lattice D is perfect, if for each finite dimension indecomposable K-linear representation $\rho_X : L \to \mathfrak{L}(X)$ over any field K, the image $\rho_X(a) \subseteq X$ of a is either zero, or $\rho_X(a) = X$, where $\mathfrak{L}(X)$ is the lattice of all vector K-subspaces of X.

Admissible elements[2] are the building blocks in the construction of perfect elements, see [GP74], [GP76], and also [St04]. Making use of certain elegant endomorphisms in D^4, Herrmann in [H82] also constructed perfect elements. The main purpose of this article is to obtain *a connection between admissible elements and Herrmann's endomorphisms.*

First, in this section, we outline the idea of admissible elements and the construction of the distributive sublattice of perfect polynomials [GP74]. We think today, that apart from being helpful in construction of perfect elements, the admissible elements are interesting in themselves.

Further, we outline properties of admissible elements obtained in this work: *finite classification*, *$\varphi-$homomorphism*, *reduction to atomic elements* and *periodicity*, see Section 1.2. In Section 1.5 we outline the connection between admissible elements and Herrmann's endomorphisms [H82]. Not only perfect elements in D^4, also the admissible elements e_α and $f_{\alpha 0}$ in D^4 can be obtained by Herrmann's endomorphisms, see Theorem 3.3.6.

In Section 2, we construct admissible elements in D^4, by applying the technique developed in [St04] for the modular lattice $D^{2,2,2}$.

In Section 3 we study Herrmann's endomorphisms used in his construction of the perfect elements in D^4. We show *how the admissible elements introduced in Section 2 are constructed by means of Herrmann's endomorphisms γ_{ij}*. Set of admissible elements e_α (resp. $f_{\alpha 0}$) (described by Table 2.3) coincide with the set

$$\{e_i a_r^{jl} a_s^{lk} a_t^{kj}\}, \text{ resp. } \{e_i a_r^{jl} a_s^{lk} a_t^{kj}(e_i a_{t+1}^{jk} + a_{s+1}^{kl} a_{r-1}^{jl})\}, \tag{1.1}$$

where a_r^{jl} are *atomic elements* (1.12), $r, s, t \in \mathbb{Z}_+$ and $\{i, j, k, l\} = \{1, 2, 3, 4\}$, see Proposition 3.1.1.

Herrmann's endomorphism γ_{ij} and the elementary map of Gelfand-Ponomarev ϕ_i act, in a sense, in opposite directions, namely the elementary map ϕ_i adds the index to the end of the admissible sequence:

$$\varphi_i \rho_{X^+}(z_\alpha) = \rho_X(z_{i\alpha}), \tag{1.2}$$

see Theorem 2.8.1, and the endomorphism γ_{ik} adds the index to the start of the admissible sequence:

$$\gamma_{ik}(z_{\alpha k}) = z_{\alpha k i}, \tag{1.3}$$

see Theorem 3.3.4.

Admissible elements e_α (similarly, $f_{\alpha 0}$) are obtained by means of Herrmann's endomorphisms as follows:

$$\gamma_{ij}^t \gamma_{ik}^s \gamma_{il}^r(e_i) = e_i a_t^{kl} a_s^{jl} a_r^{kj}, \tag{1.4}$$

where $r, s, t \in \mathbb{Z}_+$ and $\{i, j, k, l\} = \{1, 2, 3, 4\}$, see Theorem 3.3.6.

[1] Equivalently, instead of representations of D^4, one can speak about *quadruple of subspaces*, [GP70].

[2] The definition of notion *admissible* will be given shortly.

1.1. **The idea of admissible elements of Gelfand and Ponomarev.** Here, we describe, omitting some details, the idea of constructing the elementary maps and admissible elements of Gelfand-Ponomarev. As we will see below, the admissible elements *grow*, in a sense, from generators of the lattice or from the lattice's unity.

Following Bernstein, Gelfand and Ponomarev [BGP73], given a modular lattice L and a field K, we use the Coxeter functor

$$\Phi^+ : \mathrm{rep}_K L \longrightarrow \mathrm{rep}_K L,$$

see [St04, App. A] for details. Given a representation $\rho_X : L \longrightarrow \mathcal{L}(X)$ in $\mathrm{rep}_K L$, we denote by

$$\rho_{X^+} : L \longrightarrow \mathcal{L}(X^+)$$

the image representation $\Phi^+\rho_X$ in $\mathrm{rep}_K L$ under the functor Φ^+.

Construction of the elementary map φ. Let there exist a map φ mapping every subspace $A \in \mathcal{L}(X^+)$ to some subspace $B \in \mathcal{L}(X)$:

$$\begin{aligned} &\varphi : \mathcal{L}(X^+) \longrightarrow \mathcal{L}(X), \\ &\varphi A = B, \quad \text{ or } \quad A \overset{\varphi}{\longmapsto} B. \end{aligned} \tag{1.5}$$

It turns out that for many elements $a \in L$ there exists an element $b \in L$ (at least in $L = D^4$ and $D^{2,2,2}$) such that

$$\varphi(\rho_{X^+}(a)) = \rho_X(b) \tag{1.6}$$

for every pair of representations (ρ_X, ρ_{X^+}), where $\rho_{X^+} = \Phi^+\rho_X$. The map φ in (1.6) is constructed in such a way, that φ do not depend on ρ_X. Thus, we can write

$$a \overset{\varphi}{\longmapsto} b. \tag{1.7}$$

In particular, eq. (1.6) and (1.7) are true for admissible elements, whose definition will be given later, see Section 2. Eq. (1.6) is the main property characterizing the admissible elements.

For D^4, Gelfand and Ponomarev constructed 4 maps of form (1.5): $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ and called them *elementary maps*. Given a sequence of relations

$$a_{i_1} \overset{\varphi_{i_1}}{\longmapsto} a_{i_2} \overset{\varphi_{i_2}}{\longmapsto} a_{i_3} \overset{\varphi_{i_3}}{\longmapsto} \cdots \overset{\varphi_{i_{n-2}}}{\longmapsto} a_{i_{n-1}} \overset{\varphi_{i_{n-1}}}{\longmapsto} a_n, \tag{1.8}$$

we say that elements $a_{i_2}, a_{i_3}, \dots, a_{i_{n-2}}, a_{i_{n-1}}, a_{i_n}$ *grow* from the element a_{i_1}. By abuse of language, we say that elements growing from generators e_i or unity I are *admissible elements*, and the corresponding sequence of indices $\{i_n i_{n-1} i_{n-2} \dots i_2 i_1\}$ is said to be the *admissible sequence*. For details in the cases $D^{2,2,2}$ and D^4, see [St04].

1.2. **Reduction of the admissible elements to atomic elements.** Here, we briefly give steps of creation of the admissible elements in this work.

A) Finite classification. The admissible elements (and admissible sequences) can be reduced to a finite number of classes, see [St04]. Admissible sequences for D^4 are depicted on Fig. 1.1.

The key property of admissible sequences allowing to do this classification is the following relation between maps φ_i.

For $D^{2,2,2}$ (see [St04, Prop. 2.10.2]):

$$\begin{aligned} &iji = iki, \\ &\varphi_i\varphi_j\varphi_i = \varphi_i\varphi_k\varphi_i, \text{ where } \{i,j,k\} = \{1,2,3\}. \end{aligned}$$

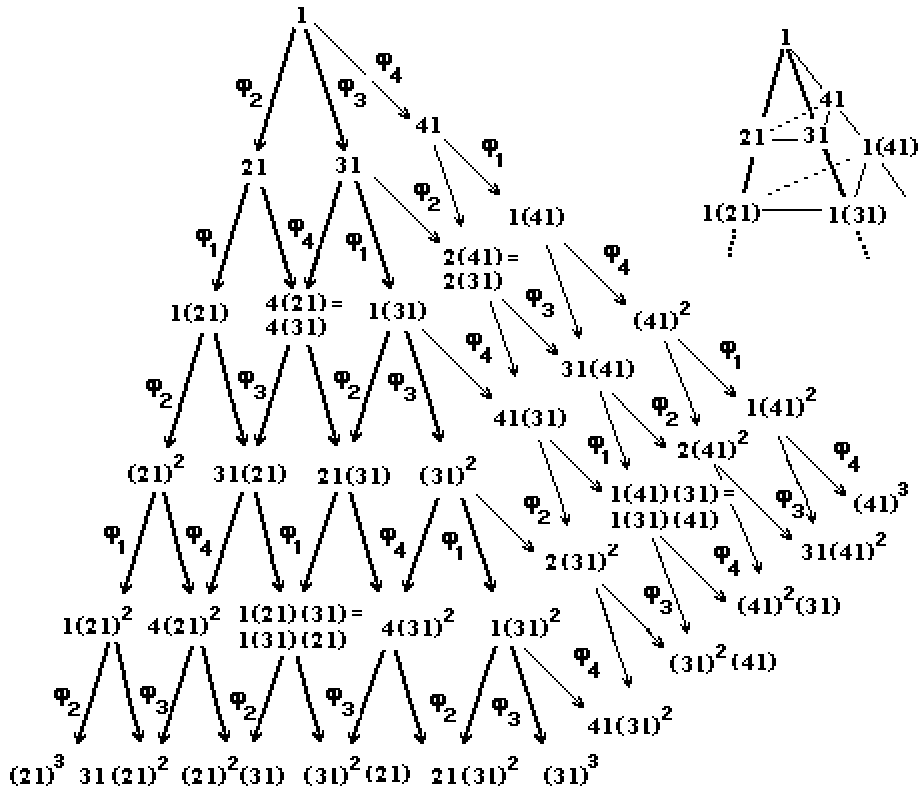

FIGURE 1.1. Action of maps φ_i on the admissible sequences for D^4

For D^4 (see [St04, Prop. 4.6.1]):

$$ikj = ilj,$$
$$\varphi_i\varphi_k\varphi_j = \varphi_i\varphi_l\varphi_j, \text{ where } \{i,j,k,l\} = \{1,2,3,4\}.$$

B) φ_i-homomorphism. In a sense, the elementary maps φ_i are homomorphic with respect to admissible elements. More exactly, introduce a notion of φ_i-homomorphic elements. Let

$$a \overset{\varphi_i}{\longmapsto} \tilde{a} \quad \text{and} \quad p \overset{\varphi_i}{\longmapsto} \tilde{p}. \tag{1.9}$$

The element $a \in L$ is said to be φ_i-*homomorphic*, if

$$ap \overset{\varphi_i}{\longmapsto} \tilde{a}\tilde{p} \tag{1.10}$$

for all $p \in L$.

C) Reduction to atomic elements. We permanently apply the following mechanism of creation of admissible elements from more simple elements. Let φ_i be any elementary map, and

$$a \overset{\varphi_i}{\longmapsto} \tilde{a}, \quad b \overset{\varphi_i}{\longmapsto} \tilde{b}, \quad c \overset{\varphi_i}{\longmapsto} \tilde{c}, \quad p \overset{\varphi_i}{\longmapsto} \tilde{p}. \tag{1.11}$$

Suppose $abcp$ is any admissible element and a, b, c are φ_i-homomorphic. By means of relation (1.10) we construct new admissible element

$$\tilde{a}\tilde{b}\tilde{c}\tilde{p},$$

The elements a, b, c are called *atomic*. For definition of atomic elements in $D^{2,2,2}$, see [St04].

For the case of D^4, *atomic* lattice polynomials a^{ij}_n, where $i, j \in \{1, 2, 3, 4\}$, $n \in \mathbb{Z}_+$, are defined as follows

$$a^{ij}_n = \begin{cases} a^{ij}_n = I & \text{for } n = 0, \\ a^{ij}_n = e_i + e_j a^{kl}_{n-1} = e_i + e_j a^{lk}_{n-1} & \text{for } n \geq 1, \end{cases} \tag{1.12}$$

where $\{i, j, k, l\}$ is the permutation of $\{1, 2, 3, 4\}$.

D) Periodicity. Admissible sequences obtained by reducing in heading A) possess some *periodicity*. The corresponding admissible elements are also periodic. On Fig. 1.2 we see some examples of admissible elements for D^4. Two vertical lines on each of three cylinders on Fig. 1.2 correspond to six series of inclusions of admissible elements:

$$\begin{aligned}
&\dots \subseteq e_{1(41)^r} \subseteq e_{1(41)^{r-1}} \subseteq \dots \subseteq e_{1(41)} \subseteq e_1, \\
&\dots \subseteq e_{(41)^r} \subseteq e_{(41)^{r-1}} \subseteq \dots \subseteq e_{(41)}, \\
&\dots \subseteq e_{1(31)^s(41)^r} \subseteq e_{1(31)^{s-1}(41)^r} \subseteq \dots \subseteq e_{1(31)(41)^r} \subseteq e_{1(41)^r}, \\
&\dots \subseteq e_{(31)^s(41)^r} \subseteq e_{(31)^{s-1}(41)^r} \subseteq \dots \subseteq e_{(31)(41)^r} \subseteq e_{(41)^r}, \\
&\dots \subseteq e_{1(21)^t(31)^s(41)^r} \subseteq e_{1(21)^{t-1}(31)^s(41)^r} \subseteq \dots \subseteq e_{1(21)(31)^s(41)^r} \subseteq e_{1(31)^s(41)^r}, \\
&\dots \subseteq e_{(21)^t(31)^s(41)^r} \subseteq e_{(21)^{t-1}(31)^s(41)^r} \subseteq \dots \subseteq e_{(21)(31)^s(41)^r} \subseteq e_{(31)^s(41)^r}.
\end{aligned} \tag{1.13}$$

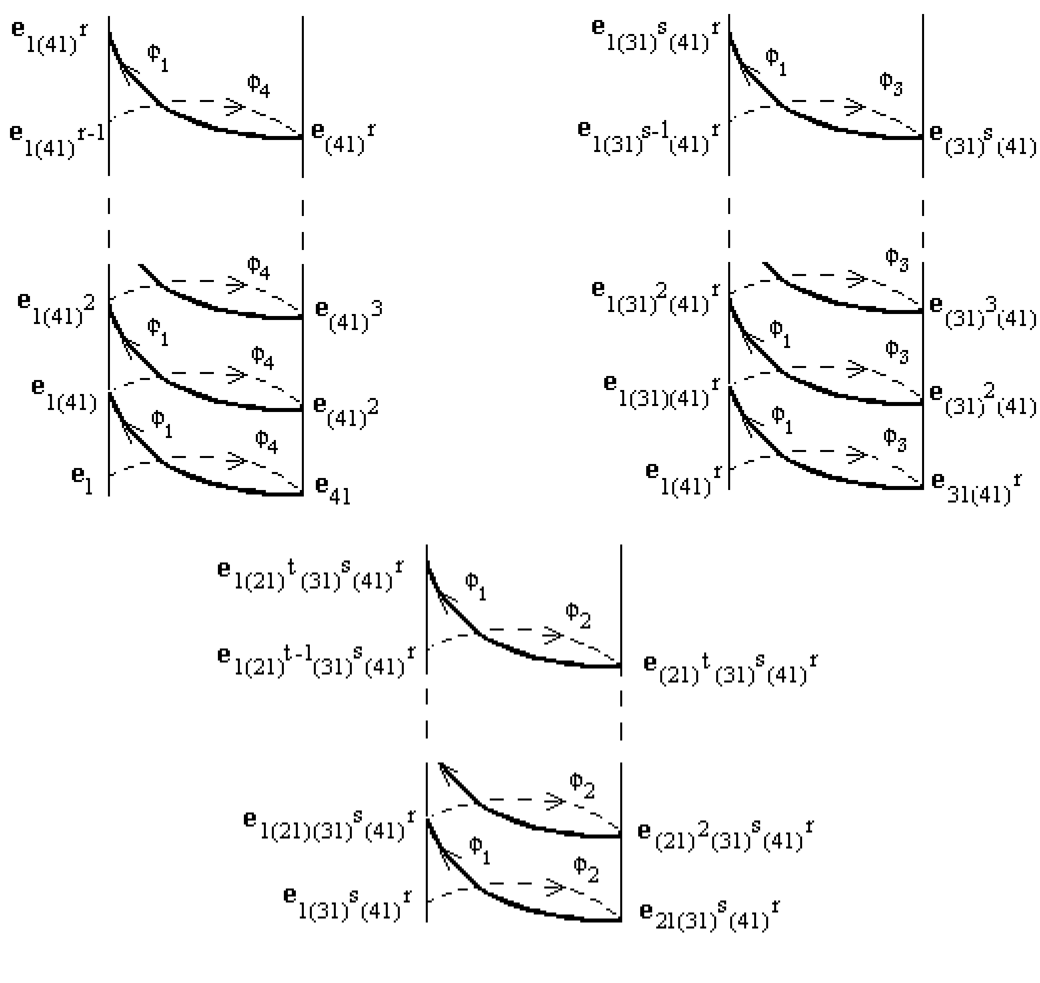

FIGURE 1.2. Periodicity of admissible elements

Three cylindric helices on Fig. 1.2 correspond to relations

$$\begin{array}{lll} e_{1(41)^{r-1}} \xmapsto{\varphi_4} e_{(41)^r}, & & e_{1(31)^{s-1}(41)^r} \xmapsto{\varphi_3} e_{(31)^s(41)^r}, \\ e_{(41)^r} \xmapsto{\varphi_1} e_{1(41)^r}, & & e_{(31)^s(41)^r} \xmapsto{\varphi_1} e_{1(31)^s(41)^r}, \\ & e_{1(21)^{t-1}(31)^s(41)^r} \xmapsto{\varphi_2} e_{(21)^t(31)^s(41)^r}, & \\ & e_{(21)^t(31)^s(41)^r} \xmapsto{\varphi_1} e_{1(21)^t(31)^s(41)^r}. & \end{array} \tag{1.14}$$

Relation (1.14) is a particular case of Theorem 2.8.1 (theorem on admissible elements; compare also with the theorem on admissible elements for $D^{2,2,2}$ [St04, Th. 2.12.1]).

1.3. **Direct construction of admissible elements.** Admissible elements for D^r in [GP74], [GP76] are built recurrently in the length of multi-indices named *admissible sequences*. In this work we suggest a direct method for creating admissible elements. For $D^{2,2,2}$ (resp. D^4), the admissible sequences and admissible elements form 14 classes (resp. 8 classes) and possess some periodicity properties, see Section 1.7.1, Tables 2.2., 2.3, (resp. Section 1.7.2, Tables 4.1, 4.3) from [St04].

For the definition of admissible polynomials in D^4, see Table 2.3.

For the definition of admissible polynomials in D^4 due to Gelfand and Ponomarev and examples obtained from this definition, see Section 2.8.2.

1.4. **Eight types of admissible sequences.** The admissible sequence for D^4 is defined as follows. Consider a finite sequence of indices $s = i_n \dots i_1$, where $i_p \in \{1, 2, 3\}$. The sequence s is said to be *admissible* if

(a) Adjacent indices are distinct ($i_p \neq i_{p+1}$).

(b) In each subsequence ijl, we can replace index j by k. In other words: $\dots ijl \dots = \dots ikl \dots$, where all indices i, j, k, l are distinct.

Any admissible sequence with $i = 1$ for D^4 may be transformed to one of the next 8 types (Proposition 2.1.3, Table 2.1):

$$\begin{array}{l} 1)\ (21)^t(41)^r(31)^s = (21)^t(31)^s(41)^r, \\ 2)\ (31)^t(41)^r(21)^s = (31)^t(21)^s(41)^r, \\ 3)\ (41)^t(31)^r(21)^s = (41)^t(21)^s(31)^r, \\ 4)\ 1(41)^t(31)^r(21)^s = 1(31)^t(41)^s(21)^r = 1(21)^t(31)^s(41)^r, \\ 5)\ 2(41)^r(31)^s(21)^t = 2(31)^{s+1}(41)^{r-1}(21)^t, \\ 6)\ 3(41)^r(21)^s(31)^t = 3(21)^{s+1}(41)^{r-1}(31)^t, \\ 7)\ 4(21)^r(31)^s(41)^t = 3(31)^{s+1}(21)^{r-1}(41)^t, \\ 8)\ (14)^r(31)^s(21)^t = (14)^r(21)^{t+1}(31)^{s-1} = (13)^s(41)^r(21)^t = \\ \quad (13)^s(21)^{t+1}(41)^{r-1} = (12)^{t+1}(41)^r(31)^{s-1} = (12)^{t+1}(31)^{s-1}(41)^r. \end{array} \tag{1.15}$$

1.5. **Admissible elements in D^4 and Herrmann's endomorphisms.** C. Herrmann introduced in [H82] the commuting endomorphisms γ_{ik} $(i, k = 1, 2, 3)$ in D^4 and used them in the construction of the perfect elements, see Section 3.

It turned out, that endomorphisms γ_{ik} are also closely connected with admissible elements. First of all, the endomorphism γ_{ik} acts on the admissible element $e_{\alpha k}$ such that

$$\gamma_{ik}(e_{\alpha k}) = e_{\alpha k i},$$

There is a *similarity between the action of Herrmann's endomorphism* γ_{ik} *and the action of the elementary map of Gelfand-Ponomarev* ϕ_i. The endomorphism γ_{ik} and the elementary map ϕ_i act, in a sense, in opposite directions, namely the endomorphism γ_{ik} adds the index to the start of the admissible sequence (see Theorem 3.3.4), and the elementary map ϕ_i adds the index to the end (see Theorem 2.8.1).

Further, every admissible element e_α (resp. $f_{\alpha 0}$) in D^4 has an unified form $e_1 a^{34}_t a^{24}_s a^{32}_r$, see Table 3.1, and every admissible element e_α (resp. $f_{\alpha 0}$) is obtained by means of the sequence of Herrmann's endomorphisms as follows:

$$\gamma^t_{12}\gamma^s_{13}\gamma^r_{14}(e_1) = e_1 a^{34}_t a^{24}_s a^{32}_r,$$

and, respectively,

$$\gamma^t_{12}\gamma^s_{13}\gamma^r_{14}(f_{10}) = e_1 a^{34}_t a^{24}_s a^{32}_r (e_1 a^{34}_{t+1} + a^{24}_{s+1} a^{32}_{r-1}),$$

see Theorem 3.3.6.

1.6. **Examples of admissible elements.** Admissible sequences and admissible elements are taken from Table 2.3.

For $n = 1$: $\alpha = 1$, (Table 2.3, Line $G11$, $r = 0, s = 0, t = 0$). We have

$$\begin{aligned} &e_1 = e_1 \quad \text{(admissible element } e_1 \text{ coincides with generator } e_1\text{)},\\ &f_{10} = e_1(e_2 + e_3 + e_4) \subseteq e_1. \end{aligned}$$

For $n = 2$: $\alpha = 21$, (Table 2.3, Line $G21$, $r = 0, s = 0, t = 1$). We have

$$\begin{aligned} e_{21} =& e_2(e_3 + e_4),\\ f_{210} =& e_2(e_3 + e_4)(e_4 + e_3(e_1 + e_2) + e_1) =\\ & e_2(e_3 + e_4)(e_4 + e_2(e_1 + e_3) + e_1) =\\ & e_2(e_3 + e_4)(e_2(e_1 + e_4) + e_2(e_1 + e_3)) \subseteq e_{21}. \end{aligned}$$

For $n = 3$: Consider two admissible sequences: $\alpha = 121$ and $\alpha = 321 = 341$.

1) $\alpha = 121$, (Table 2.3, Line $G11$, $r = 0, s = 0, t = 1$). We have

$$\begin{aligned} e_{121} =& e_1 a^{34}_2 = e_1(e_3 + e_4(e_1 + e_2)) =\\ & e_1(e_3(e_1 + e_2) + e_4(e_1 + e_2)),\\ f_{1210} =& e_{121}(e_1 a^{32}_1 + a^{24}_1 a^{34}_1) =\\ & e_{121}(e_1(e_2 + e_3) + (e_2 + e_4)(e_3 + e_4)) =\\ & e_1(e_3 + e_4(e_1 + e_2))(e_1(e_2 + e_3) + (e_2 + e_4)(e_3 + e_4)) \subseteq e_{121}. \end{aligned}$$

2) $\alpha = 321 = 341$, (Table 2.3, Line $G31$, $t = 0, s = 0, r = 1$). We have

$$\begin{aligned} e_{321} = e_{341} =& e_3 a^{21}_1 a^{14}_1 =\\ & e_3(e_2 + e_1)(e_4 + e_1),\\ f_{3210} = f_{3410} =& e_{321}(a^{14}_2 + e_3 a^{24}_1) =\\ & e_{321}(e_1 + e_4(e_3 + e_2) + e_3(e_2 + e_4)) =\\ & e_{321}(e_1 + (e_3 + e_2)(e_4 + e_2)(e_3 + e_4)) =\\ & e_3(e_2 + e_1)(e_4 + e_1)(e_1 + (e_3 + e_2)(e_4 + e_2)(e_3 + e_4)) \subseteq e_{321}. \end{aligned}$$

For $n=4$: $\alpha = 2341 = 2321 = 2141$, (Table 2.3, Line $F21$, $s=0, t=1, r=1$). We have

$$\begin{aligned} e_{2141} =& e_2 a_2^{41} a_1^{34} = \\ & e_2(e_4 + e_1(e_3+e_2))(e_3+e_4), \\ f_{21410} =& e_{2141}(a_2^{34} + e_1 a_2^{24}) = \\ & e_{2141}(e_2(e_4+e_1(e_3+e_2)) + e_1(e_2+e_4(e_1+e_3))) \subseteq e_{21}. \end{aligned}$$

1.7. **Cumulative polynomials.** According to Gelfand and Ponomarev we consider elements $e_t(n)$ and $f_0(n)$ used for construction of the perfect elements, [GP74, p.7, p.53].

The *cumulative elements* $e_t(n)$, where $t = 1,2,3,4$ and $f_0(n)$ are sums of all admissible elements of the same length n, where n is the length of the multi-index. So, for the case D^4, the cumulative polynomials defined as follows:

$$\begin{aligned} e_t(n) &= \sum e_{i_n \dots i_2 t}, \qquad t = 1,2,3,4, \\ f_0(n) &= \sum f_{i_n \dots i_2 0}, \end{aligned} \tag{1.16}$$

where admissible elements e_α and $f_{\alpha 0}$ are defined in Table 2.3, see also Section 1.4, Section 1.6.

Examples of the cumulative polynomials in D^4.

For $n=1$:

$$\begin{aligned} e_1(1) &= e_1, \\ f_0(1) &= I, \end{aligned}$$

i.e., the *cumulative polynomials* for $n=1$ coincide with the generators of D^4.

For $n=2$: We use here the following property:

$$e_\alpha(e_1 a_t^{34} + a_s^{24} a_r^{32}) = e_\alpha(a_t^{43} + e_2 a_{s-1}^{41} a_{r+1}^{31}),$$

see Lemma 2.8.2, heading 2). For the type $G11$ in Table 2.3, we have

$$e_\alpha(e_1 a_{2t+1}^{34} + a_{2s+1}^{24} a_{2r-1}^{32}) = e_\alpha(a_{2t+1}^{43} + e_2 a_{2s}^{41} a_{2r}^{31}), \tag{1.17}$$

Taking in (1.17), $r = s = t = 0$ we have

$$f_{10} = e_1(a_1^{43} + e_2) = e_1(e_2 + e_3 + e_4).$$

Thus,

$$\begin{aligned} e_1(2) =& e_{21} + e_{31} + e_{41} = \\ & e_2(e_3+e_4) + e_3(e_2+e_4) + e_4(e_2+e_3), \\ f_0(2) =& f_{10} + f_{20} + f_{30} + f_{40} = \\ & e_1(e_2+e_3+e_4) + e_2(e_1+e_3+e_4) + e_3(e_1+e_2+e_4) + e_4(e_1+e_2+e_3). \end{aligned} \tag{1.18}$$

For $n=3$:

$$e_1(3) = e_{321} + e_{231} + e_{341} + e_{431} + e_{241} + e_{421},$$

$$\begin{aligned} f_0(3) =& (f_{120} + f_{130} + f_{140}) + (f_{210} + f_{230} + f_{240}) + \\ & (f_{310} + f_{320} + f_{340}) + (f_{310} + f_{320} + f_{340}), \end{aligned}$$

and so forth, see also Section 1.6.

1.8. **Perfect elements.**

1.8.1. *Perfect cubes in the free modular lattice D^r.* In [GP74], Gelfand and Ponomarev constructed the sublattice B of perfect elements for the free modular lattice D^r with r generators:

$$B = B^+ \bigcup B^+, \quad \text{where } B^+ = \bigcup_{n=1}^{\infty} B^+(n), \quad B^- = \bigcup_{n=1}^{\infty} B^-(n).$$

They proved that every sublattice $B^+(n)$ (resp. $B^-(n)$) is 2^r-element Boolean algebra, so-called *Boolean cube* (which can be also named *perfect Boolean cube*) and these *cubes* are ordered in the following way. Every element of the cube $B^+(n)$ is included in every element of the cube $B^+(n+1)$, i.e.,

$$\left.\begin{array}{l} v^+(n) \in B^+(n) \\ v^+(n+1) \in B^+(n+1) \end{array}\right\} \Longrightarrow v^+(n+1) \subseteq v^+(n).$$

By analogy, the dual relation holds:

$$\left.\begin{array}{l} v^-(n) \in B^-(n) \\ v^-(n+1) \in B^-(n+1) \end{array}\right\} \Longrightarrow v^-(n) \subseteq v^-(n+1).$$

1.8.2. *Perfect elements in D^4.* Perfect elements in D^4 (similarly, for D^r, see [GP74, p.6]) are constructed by means of cumulative elements (1.16) as follows:

$$h_t(n) = \sum_{j \neq t} e_j(n). \tag{1.19}$$

Elements (1.19) generate perfect Boolean cube $B^+(n)$ from Section 1.8.1, see [GP74].

Examples of perfect elements.

For $n = 1$:

$$\begin{aligned} h_1(1) &= e_2 + e_3 + e_4, \quad h_2(1) = e_1 + e_3 + e_4, \\ h_3(1) &= e_1 + e_2 + e_4, \quad h_4(1) = e_1 + e_2 + e_3. \end{aligned} \tag{1.20}$$

For $n = 2$, by (1.18) we have

$$\begin{aligned} h_1(2) = & e_2(2) + e_3(2) + e_4(2) = \\ & (e_{12} + e_{32} + e_{42}) + (e_{13} + e_{23} + e_{43}) + (e_{14} + e_{24} + e_{34}) = \\ & e_1(e_3 + e_4) + e_3(e_1 + e_4) + e_4(e_1 + e_3) + \\ & e_1(e_2 + e_4) + e_2(e_1 + e_4) + e_4(e_1 + e_2) + \\ & e_1(e_2 + e_3) + e_2(e_1 + e_3) + e_3(e_1 + e_2) = \\ & (e_1 + e_3)(e_1 + e_4)(e_3 + e_4) + (e_1 + e_2)(e_1 + e_4)(e_2 + e_4) + \\ & (e_1 + e_2)(e_1 + e_3)(e_2 + e_3). \end{aligned} \tag{1.21}$$

Let $h^{\max}(n)$ (resp. $h^{\min}(n)$) be the maximal (resp. minimal) element in the cube $B^+(n)$.

$$\begin{aligned} h^{\max}(n) &= \sum_{i=1,2,3,4} h_i(n), \\ h^{\min}(n) &= \bigcap_{i=1,2,3,4} h_i(n). \end{aligned} \tag{1.22}$$

1.8.3. *Perfect elements in D^4 by C. Herrmann.* C. Herrmann introduced in [H82], [H84] polynomials q_{ij} and associated endomorphisms γ_{ij} of D^4 playing the central role in his construction of perfect polynomials. For $\{i,j,k,l\} = \{1,2,3,4\}$, define

$$q_{ij} = q_{ji} = q_{kl} = q_{lk} = (e_i + e_j)(e_k + e_l),$$

and

$$\gamma_{ij} f(e_1, e_2, e_3, e_4) = f(e_1 q_{ij}, e_2 q_{ij}, e_3 q_{ij}, e_4 q_{ij}),$$

see Section 3.3.

C. Herrmann's construction of perfect elements s_n, t_n and $p_{i,n}$, where $i = 1,2,3,4$, is as follows:

$$\begin{aligned} s_0 = I, \quad & s_1 = e_1 + e_2 + e_3 + e_4, \\ & s_{n+1} = \gamma_{12}(s_n) + \gamma_{13}(s_n) + \gamma_{14}(s_n), \end{aligned}$$

$$\begin{aligned} t_0 = I, \quad & t_1 = (e_1 + e_2 + e_3)(e_1 + e_2 + e_4)(e_1 + e_3 + e_4)(e_2 + e_3 + e_4), \\ & t_{n+1} = \gamma_{12}(t_n) + \gamma_{13}(t_n) + \gamma_{14}(t_n), \end{aligned}$$

$$\begin{aligned} p_{i,0} = I, \quad & p_{i,1} = e_i + t_1, \text{ where } i = 1,2,3,4, \\ & p_{i,n+1} = \gamma_{ij}(p_{j,n}) + \gamma_{ik}(p_{k,n}) + \gamma_{il}(p_{l,n}), \\ & \quad \text{where } i = 1,2,3,4, \text{ and } \{i,j,k,l\} = \{1,2,3,4\}, \end{aligned}$$

see Section 3.4.2. Then,

$$\begin{aligned} s_n &= \sum_{r+s+t=n-1} \gamma_{12}^t \gamma_{13}^s \gamma_{14}^r \Big(\sum_{i=1,2,3,4} e_i \Big) = \sum_{i=1,2,3,4} e_i(n) \simeq h^{\max}(n), \\ t_n &= \sum_{r+s+t=n} \gamma_{12}^t \gamma_{13}^s \gamma_{14}^r \Big(\sum_{i=1,2,3,4} e_i(e_j + e_k + e_l) \Big) = f_0(n+1) \simeq h^{\min}(n), \qquad (1.23) \\ p_{i,n} &= e_i(n) + f_0(n+1) \simeq h_j(n) h_k(n) h_l(n), \text{ for } \{i,j,k,l\} = \{1,2,3,4\}, \end{aligned}$$

where $a \simeq b$ means modulo linear equivalence, see Theorem 3.4.3, Theorem 3.4.4. Thus, the lattice of perfect elements generated by elements s_n, t_n, $p_{i,n}$ introduced by Herrmann [H82] coincide modulo linear equivalence with the Gelfand-Ponomarev perfect elements, see Table 1.1.

Table 1.1 gives two system of generators in the sublattice of the perfect elements $B^+(n)$ in D^4: coatoms $h_i(n)$ (by Gelfand-Ponomarev) vs. atoms $p_{i,n}$ (by C. Herrmann).

2. **Atomic and admissible polynomials**

2.1. **Admissible sequences.** For definition of admissible sequences in the case of the modular lattice D^4, see Section 1.4. Essentially, the fundamental property of this definition is

$$ijk = ilk \quad \text{for all } \{i,j,k,l\} = \{1,2,3,4\}. \qquad (2.1)$$

Relation (2.1) is our main tool in all further calculations of admissible sequences of D^4.

Without loss of generality only sequences starting at 1 can be considered. The following proposition will be used for the classification of admissible sequences in D^4.

N	Gelfand - Ponomarev definition	Herrmann definition	Sum of cumulative polynomials
1	$\sum\limits_{i=1,2,3,4} h_i(n)$	s_n	$\sum\limits_{i=1,2,3,4} e_i(n)$
2	$h_1(n)$	$p_{2,n} + p_{3,n} + p_{4,n}$	$e_2(n) + e_3(n) + e_4(n)$
3	$h_2(n)$	$p_{1,n} + p_{3,n} + p_{4,n}$	$e_1(n) + e_3(n) + e_4(n)$
4	$h_3(n)$	$p_{1,n} + p_{2,n} + p_{4,n}$	$e_1(n) + e_2(n) + e_4(n)$
5	$h_4(n)$	$p_{1,n} + p_{2,n} + p_{3,n}$	$e_1(n) + e_2(n) + e_3(n)$
6	$h_1(n)h_2(n)$	$p_{3,n} + p_{4,n}$	$e_3(n) + e_4(n) + f_0(n+1)$
7	$h_1(n)h_3(n)$	$p_{2,n} + p_{4,n}$	$e_2(n) + e_4(n) + f_0(n+1)$
8	$h_1(n)h_4(n)$	$p_{2,n} + p_{3,n}$	$e_2(n) + e_3(n) + f_0(n+1)$
9	$h_2(n)h_3(n)$	$p_{1,n} + p_{4,n}$	$e_1(n) + e_4(n) + f_0(n+1)$
10	$h_2(n)h_4(n)$	$p_{1,n} + p_{3,n}$	$e_1(n) + e_3(n) + f_0(n+1)$
11	$h_3(n)h_4(n)$	$p_{1,n} + p_{2,n}$	$e_1(n) + e_2(n) + f_0(n+1)$
12	$h_1(n)h_2(n)h_3(n)$	$p_{4,n}$	$e_4(n) + f_0(n+1)$
13	$h_1(n)h_2(n)h_4(n)$	$p_{3,n}$	$e_3(n) + f_0(n+1)$
14	$h_1(n)h_3(n)h_4(n)$	$p_{2,n}$	$e_2(n) + f_0(n+1)$
15	$h_2(n)h_3(n)h_4(n)$	$p_{1,n}$	$e_1(n) + f_0(n+1)$
16	$\bigcap\limits_{i=1,2,3,4} h_i(n)$	t_n	$f_0(n+1) = f_0^\vee(n+1)$

TABLE 1.1. Perfect elements in $B^+(n)$

Proposition 2.1.1. *The following relations hold*

1) $(31)^r(32)^s(31)^t = (32)^s(31)^{r+t}$,
2) $(31)^r(21)^s(31)^t = (31)^{r+t}(21)^s$,
3) $(42)^r(41)^s = (41)^r(31)^s, \quad s \geq 1$,
4) $2(41)^r(31)^s = 2(31)^{s+1}(41)^{r-1}, \quad r \geq 1$,
5) $(43)^r(42)^s(41)^t = (41)^r(21)^s(31)^t$,
6) $1(41)^r(21)^s = 1(21)^s(41)^r, \quad 1(i1)^r(j1)^s = 1(j1)^s(i1)^r, \quad i, j \in \{2,3,4\},\ i \neq j$,
7) $(41)^r(21)^t(31)^s = (41)^r(31)^s(21)^t$,
8) $(13)^s(21)^r = (12)^r(31)^s$,
9) $12(41)^r(31)^s(21)^t = (14)^r(31)^{s+1}(21)^t = (14)^r(21)^{t+1}(31)^s$,
10) $12(14)^r(31)^s(21)^t = (14)^r(31)^s(21)^{t+1}$,
11) $13(14)^r(31)^s(21)^t = (14)^r(31)^{s+2}(21)^{t-1}$,
12) $32(14)^r(31)^s(21)^t = (31)^s(21)^{t+1}(41)^r = 34(14)^r(31)^s(21)^t$,
13) $42(14)^r(31)^s(21)^t = (41)^s(21)^{t+1}(31)^r = 43(14)^r(31)^s(21)^t$,
14) $23(14)^r(31)^s(21)^t = (21)^{t+1}(31)^s(41)^r = 24(14)^r(31)^s(21)^t$,
15) $2(41)^r(31)^s(21)^t = 2(14)^r(31)^{s+1}(21)^{t-1}, \quad t \geq 1, s > 1$.

For the proof, see [St04, Prop. 4.1.1]. □

	Admissible Sequence	Action φ_1	Action φ_2	Action φ_3	Action φ_4
$F21$	$(21)^t(41)^r(31)^s = (21)^t(31)^s(41)^r$	$G11$	-	$G31$	$G41$
$F31$	$(31)^s(41)^r(21)^t = (31)^s(21)^t(41)^r$	$G11$	$G21$	-	$G41$
$F41$	$(41)^r(31)^s(21)^t = (41)^r(21)^t(31)^s$	$G11$	$G21$	$G31$	-
$G11$	$1(41)^r(31)^s(21)^t =$ $1(31)^s(41)^r(21)^t =$ $1(21)^t(31)^s(41)^r$	-	$F21$	$F31$	$F41$
$G21$	$2(41)^r(31)^s(21)^t = 2(31)^{s+1}(41)^{r-1}(21)^t =$ $2(14)^r(31)^{s+1}(21)^{t-1}$	$H11$	-	$F31$	$F41$
$G31$	$3(41)^r(21)^t(31)^s = 3(21)^{t+1}(41)^{r-1}(31)^s =$ $3(14)^r(21)^{s+1}(31)^{t-1}$	$H11$	$F21$	-	$F41$
$G41$	$4(21)^t(31)^s(41)^r = 4(31)^{s+1}(21)^{t-1}(41)^r =$ $4(12)^r(31)^{s+1}(41)^{t-1}$	$H11$	$F21$	$F31$	-
$H11$	$(14)^r(31)^s(21)^t = (14)^r(21)^{t+1}(31)^{s-1} =$ $(13)^s(41)^r(21)^t = (13)^s(21)^{t+1}(41)^{r-1} =$ $(12)^{t+1}(41)^r(31)^{s-1} = (12)^{t+1}(31)^{s-1}(41)^r$	-	$G21$	$G31$	$G41$

TABLE 2.1. Admissible sequences for the modular lattice D^4

Remark 2.1.2 (Note to Table 2.1)**.** 1) Type Fij (resp. Gij, $H11$) denotes the admissible sequence starting at j and ending at i. Sequences of type Fij and $H11$ contain an even number of symbols, sequences of type Gij and $H11$ contain an odd number of symbols. For differences in types $Fij, Gij, H11$, see the table.

2) Thanks to heading 15) of Proposition 2.1.1 types $H21, H31, H41$ from [St04, p. 55, Table 4.1] are excluded, and the number of different cases of admissible sequences is equal to 8 instead 11 in [St04].

Proposition 2.1.3. *Full list of admissible sequences starting at* 1 *is given by Table 2.1.*

Proof. It suffices to prove that maps φ_i, where $i = 1, 2, 3, 4$, do not lead out of Table 2.1. The exponents r, s, t may be any non-negative integer number. The proof is based on the relations from Proposition 2.1.1, see [St04, Prop. 4.1.2].

The pyramid on the Fig. 1.1 has internal points. We consider the slice $S(n)$ containing all sequences of the same length n. The slices $S(3)$ and $S(4)$ are shown on the Fig. 2.1. The slice $S(4)$ contains only one internal point

$$14(21) = 13(21) = 13(41) = 12(41) = 14(31) = 12(31). \tag{2.2}$$

The slices $S(4)$ and $S(5)$ are shown on the Fig. 2.2. The slice $S(5)$ contains 3 internal points

$$\begin{aligned} &2(31)(21) = 2(41)(21), \\ &3(21)(31) = 3(41)(31), \\ &4(21)(41) = 4(31)(41). \end{aligned} \tag{2.3}$$

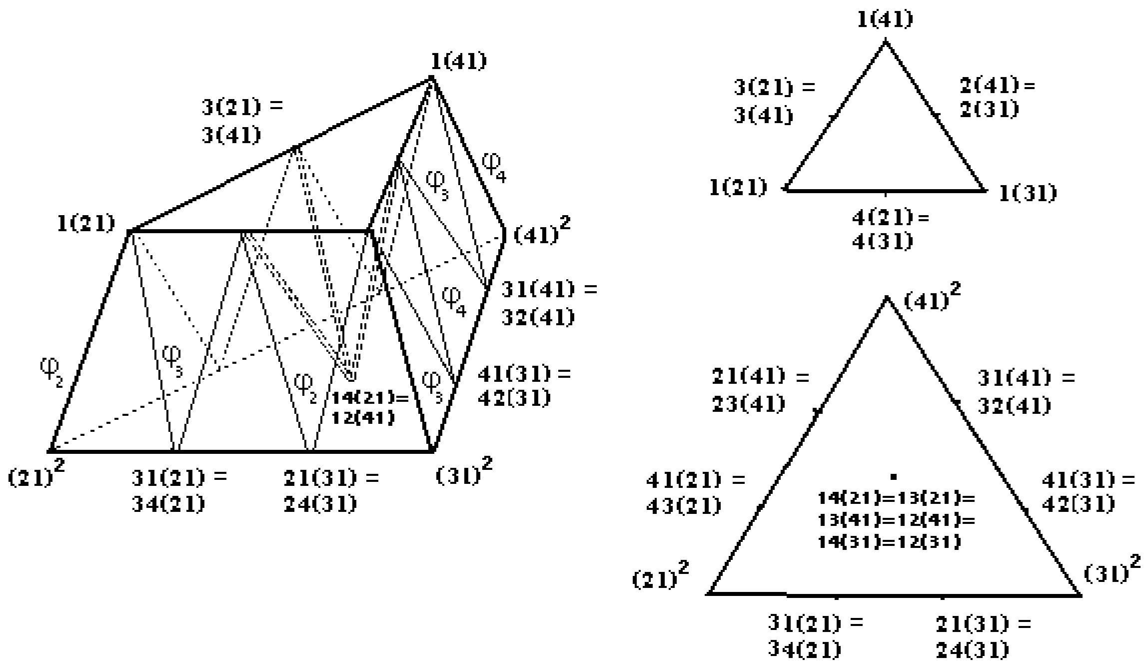

FIGURE 2.1. Slices of admissible sequences, $l = 3$ and $l = 4$

Remark 2.1.4. The slice $S(n)$ contains $\dfrac{1}{2}n(n+1)$ different admissible sequences. Actions of φ_i, where $i = 1, 2, 3, 4$ move every line in the triangle $S(n)$, which is parallel to some edge of the triangle, to the edge of $S(n+1)$. The edge containing k points is moved to $k+1$ points in the $S(n+1)$.

2.2. **Atomic polynomials and elementary maps.** The free modular lattice D^4 is generated by 4 generators:

$$D^4 = \{e_1, e_2, e_3, e_4\}.$$

Proposition 2.2.1. *1) The following property of the atomic elements take place*

$$e_j a_n^{kl} = e_j a_n^{lk} \text{ for } n \geq 1, \text{ and distinct indices } j, k, l. \tag{2.4}$$

2) The definition of the atomic elements a_n^{ij} in (1.12) is well-defined.

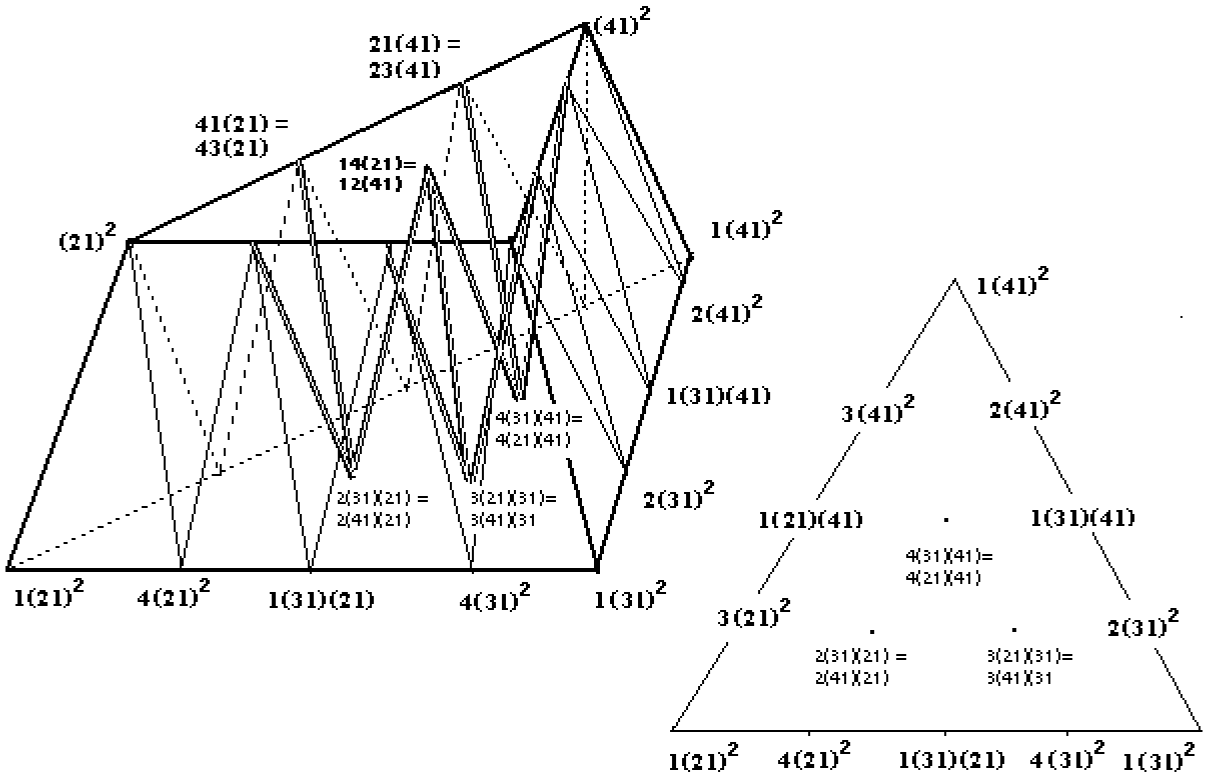

FIGURE 2.2. Slices of admissible sequences, $l = 4$ and $l = 5$

3) We have

$$a_n^{ij} \subseteq a_{n-1}^{ij} \subseteq \dots \subseteq a_2^{ij} \subseteq a_1^{ij} \subseteq a_0^{ij} = I \text{ for all } i \neq j. \tag{2.5}$$

4) To equalize the lower indices of the admissible polynomials $f_{\alpha 0}$ (see Table 2.3 and Theorem 2.8.1) we will use the following relation:

$$e_j + e_i a_{t+1}^{jk} a_{s-1}^{lk} = e_j + e_k a_t^{ij} a_s^{il} \quad \text{for all } \{i, j, k, l\} = \{1, 2, 3, 4\}. \tag{2.6}$$

For the proof, see [St04, Prop. 4.2.1] □

Now we briefly recall definitions due to Gelfand and Ponomarev [GP74], [GP76], [GP77] of spaces G_i, $G_i{}'$, representations ν^0, ν^1, joint maps ψ_i, and elementary maps φ_i, where $i = 1, 2, 3, 4$. To compare these definitions with a case of the modular lattice $D^{2,2,2}$, see [St04, Sect. 2]

We denote by

$$\{Y_1, Y_2, Y_3, Y_4 \mid Y_i \subseteq X_0, i = 1, 2, 3, 4\}$$

the representation ρ_X of D^4 in the finite dimensional vector space $X = X_0$, and by

$$\{Y_1^1, Y_2^1, Y_3^1, Y_4^1 \mid Y_i^1 \subseteq X_0^1, i = 1, 2, 3, 4\}$$

the representation ρ_{X^+} of D^4. Here Y_i (resp. Y_i^1) is the image of the generator e_i under the representation ρ_X (resp. ρ_{X^+}).

$$\begin{tikzcd} & Y_3 \arrow[d, hook] & \\ Y_1 \arrow[r, hook] & X_0 & Y_2 \arrow[l, hook'] \\ \rho_X & Y_4 \arrow[u, hook] & \end{tikzcd} \qquad \begin{tikzcd} & Y_3^1 \arrow[d, hook] & \\ Y_1^1 \arrow[r, hook] & X_0^1 & Y_2^1 \arrow[l, hook'] \\ \rho_{X^+} & Y_4^1 \arrow[u, hook] & \end{tikzcd} \tag{2.7}$$

The space $X^+ = X_0^1$ — the space of the representation ρ_{X^+} — is

$$X_0^1 = \{(\eta_1, \eta_2, \eta_3, \eta_4) \mid \eta_i \in Y_i, \quad \sum \eta_i = 0\},$$

where $i \in \{1, 2, 3, 4\}$. We set

$$R = \bigoplus_{i\,=\,1,2,3,4} Y_i,$$

i.e.,

$$R = \{(\eta_1, \eta_2, \eta_3, \eta_4) \mid \eta_i \in Y_i, \quad i = 1, 2, 3\}.$$

Then, $X_0^1 \subseteq R$.

For the case of D^4, the spaces G_i and G'_i are introduced as follows:

$$\begin{aligned} G_1 &= \{(\eta_1, 0, 0, 0) \mid \eta_1 \in Y_1\}, & G_1' &= \{(0, \eta_2, \eta_3, \eta_4) \mid \eta_i \in Y_i\}, \\ G_2 &= \{(0, \eta_2, 0, 0) \mid \eta_2 \in Y_2\}, & G_2' &= \{(\eta_1, 0, \eta_3, \eta_4) \mid \eta_i \in Y_i\}, \\ G_3 &= \{(0, 0, \eta_3, 0) \mid \eta_3 \in Y_3\}, & G_3' &= \{(\eta_1, \eta_2, 0, \eta_4) \mid \eta_i \in Y_i\}, \\ G_4 &= \{(0, 0, 0, \eta_4) \mid \eta_4 \in Y_4\}, & G_4' &= \{(\eta_1, \eta_2, \eta_3, 0) \mid \eta_i \in Y_i\}. \end{aligned} \tag{2.8}$$

For details, see [GP74, p.43].

The associated representations ν_0, ν_1 in R are defined by Gelfand and Ponomarev [GP74, eq.(7.2)]:

$$\begin{aligned} \nu^0(e_i) &= X_0^1 + G_i, \quad i = 1, 2, 3, \\ \nu^1(e_i) &= X_0^1 G_i', \quad i = 1, 2, 3. \end{aligned} \tag{2.9}$$

Following [GP74], we introduce the elementary maps φ_i:

$$\varphi_i : X_0^1 \longrightarrow X_0, (\eta_1, \eta_2, \eta_3, \eta_4) \longmapsto \eta_i.$$

From the definition we have

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = 0. \tag{2.10}$$

2.3. **Basic relations for elementary and joint maps.** We define *joint* maps $\psi_i \colon D^4 \longrightarrow \mathcal{L}(R)$ as in the case of $D^{2,2,2}$:

$$\psi_i(a) = X_0^1 + G_i(G_i' + \nu^1(a)). \tag{2.11}$$

Proposition 2.3.1. *In the case D^4 the joint maps ψ_i satisfy the following basic relations:*

(1) $\psi_i(e_i) = X_0^1$,

(2) $\psi_i(e_j) = \nu^0(e_i(e_k + e_l))$,

(3) $\psi_i(I) = \nu^0(e_i(e_j + e_k + e_l))$,

(4) $\psi_i(e_k e_l) = \psi_j(e_k e_l) = \nu^0(e_i e_j)$.

Notions	$D^{2,2,2}$	D^4
Generators	$\{x_1 \subseteq y_1, x_2 \subseteq y_2, x_3 \subseteq y_3\}$	$\{e_1, e_2, e_3, e_4\}$
Atomic elements	$a_n^{ij} = x_i + y_j a_{n-1}^{jk}$, $A_n^{ij} = y_i + x_j A_{n-1}^{ki}$, $\{i, j, k\} = \{1, 2, 3\}$	$a_n^{ij} = e_i + e_j a^{kl}$, $\{i, j, k, l\} = \{1, 2, 3, 4\}$
Representation ρ_X	$\rho_X(x_i) = X_i, \rho_X(y_i) = Y_i$	$\rho_X(e_i) = Y_i$
Space X_0^1	$X_0^1 = \{(\eta_1, \eta_2, \eta_3) \mid \eta_i \in Y_i\}$, where $\eta_1 + \eta_2 + \eta_3 = 0$	$X_0^1 = \{(\eta_1, \eta_2, \eta_3, \eta_4) \mid \eta_i \in Y_i\}$, where $\eta_1 + \eta_2 + \eta_3 + \eta_4 = 0$
Representation ρ_{X^+}	$\rho_{X^+}(x_i) = X_i^1, \rho_{X^+}(y_i) = Y_i^1$	$\rho_{X^+}(y_i) = Y_i^1$
Spaces G_i, H_i	$G_1 = \{(\eta_1, 0, 0) \mid \eta_1 \in Y_1\}$, $H_1 = \{(\xi_1, 0, 0) \mid \xi_1 \in X_1\}$, $G_2 = \{(0, \eta_2, 0) \mid \eta_2 \in Y_2\}$, $H_2 = \{(0, \xi_2, 0) \mid \xi_2 \in X_2\}$, $G_3 = \{(0, 0, \eta_3) \mid \eta_3 \in Y_3\}$, $H_3 = \{(0, 0, \xi_3) \mid \xi_3 \in X_3\}$,	$G_1 = \{(\eta_1, 0, 0, 0) \mid \eta_1 \in Y_1\}$ $G_2 = \{(0, \eta_2, 0, 0) \mid \eta_2 \in Y_2\}$ $G_3 = \{(0, 0, \eta_3, 0) \mid \eta_3 \in Y_3\}$ $G_4 = \{(0, 0, 0, \eta_4) \mid \eta_4 \in Y_4\}$
Joint maps ψ_i	$\psi_i(a) = X_0^1 + G_i(H_i' + \nu^1(a))$	$\psi_i(a) = X_0^1 + G_i(G_i' + \nu^1(a))$
Quasi-multiplicativity	$\psi_i(a)\psi_i(b) =$ $\psi_i((a + e_i)(b + x_j x_k))$	$\psi_i(a)\psi_i(b) =$ $\psi_i((a + e_i)(b + e_j e_k e_l)$
Elementary maps φ_i	$\varphi_i : X_0^1 \longrightarrow X_0$, $(\eta_1, \eta_2, \eta_3) \longmapsto \eta_i$	$\varphi_i : X_0^1 \longrightarrow X_0$, $(\eta_1, \eta_2, \eta_3, \eta_4) \longmapsto \eta_i$
Fundamental properties of elementary maps φ_i	$\varphi_i\varphi_j\varphi_i + \varphi_i\varphi_k\varphi_i = 0$, $\varphi_i^3 = 0$	$\varphi_i\varphi_k\varphi_j + \varphi_i\varphi_l\varphi_j = 0$ $\varphi_i^2 = 0$
Fundamental properties of indices (admissible sequences)	$\boxed{iji = iki}$	$\boxed{ikj = ilj}$

TABLE 2.2. Comparison of notions in $D^{2,2,2}$ and D^4

Proof. 1) From (2.11) and (2.9) we have $\psi_i(e_i) = X_0^1 + G_i G'_i = X_0^1$.

2) We have $\psi_i(y_j) = X_0^1 + G_i(G'_i + X_0^1 G'_j)$. From $G_i \subseteq G'_j$ for $i \neq j$ and by the permutation property [St04, Sect. 2], we get

$$\psi_i(y_j) = X_0^1 + G_i(G'_i G'_j + X_0^1).$$

Since $G'_i G'_j = G_k + G_l$, where i, j, k, l are distinct indices, we have

$$\begin{aligned} \psi_i(y_j) =& X_0^1 + G_i(X_0^1 + G_k + G_l) = \\ & (X_0^1 + G_i)(X_0^1 + G_k + G_l) = \nu^0(e_i(e_k + e_l)). \end{aligned}$$

3) Again,

$$\begin{aligned} \psi_i(I) =& X_0^1 + G_i(G'_i + X_0^1) = \\ & X_0^1 + G_i(G_j + G_k + G_l + X_0^1) = \nu^0(e_i(e_j + e_k + e_l)). \end{aligned} \tag{2.12}$$

4) Since $G_i \subseteq G'_k$ for all $i \neq k$, we have

$$\begin{aligned} \psi_i(e_k e_l) =& X_0^1 + G_i(G'_i + X_0^1 G'_k G'_l) = \\ & X_0^1 + G_i(G'_i G'_k G'_l + X_0^1). \end{aligned} \tag{2.13}$$

Since $G_j(G_i + G_k + G_l) = 0$, then

$$G'_i G'_j = G_k + G_l. \tag{2.14}$$

Indeed,

$$\begin{aligned} G'_i G'_j =& (G_j + G_k + G_l)(G_i + G_k + G_l) = \\ & G_k + G_l + G_j(G_i + G_k + G_l) = G_k + G_l. \end{aligned} \tag{2.15}$$

From (2.13) and (2.14), we see that

$$\begin{aligned} \psi_i(e_k e_l) =& X_0^1 + G_i(G'_i G'_k G'_l + X_0^1) = \\ & X_0^1 + G_i(X_0^1 + G_j) = \nu_0(e_i e_j). \quad \square \end{aligned}$$

The main relation between the elementary map φ_i and the joint map ψ_i (see [St04, Prop. 2.4.3]) holds also for the case of D^4. Namely, let $a, b, c \subseteq D^4$, then

$$\begin{array}{ll} \text{(i)} & \text{If } \psi_i(a) = \nu^0(b), \text{ then } \varphi_i \rho_{X^+}(a) = \rho_X(b). \\ \text{(ii)} & \text{If } \psi_i(a) = \nu^0(b) \text{ and } \psi_i(ac) = \psi_i(a)\psi_i(c), \text{ then} \\ & \varphi_i \rho_{X^+}(ac) = \varphi_i \rho_{X^+}(a) \varphi_i \rho_{X^+}(c). \end{array} \tag{2.16}$$

From Proposition 2.3.1 and eq.(2.16) we have

Corollary 2.3.2. *For the elementary map φ_i the following basic relations hold:*

(1) $\varphi_i \rho_{X^+}(e_i) = 0$,

(2) $\varphi_i \rho_{X^+}(e_j) = \rho_X(e_i(e_k + e_l))$,

(3) $\varphi_i \rho_{X^+}(I) = \rho_X(e_i(e_j + e_k + e_l))$,

(4) $\varphi_i \rho_{X^+}(e_k y_l) = \rho_X(e_i e_j)$.

$\square$

2.4. **Additivity and multiplicativity of the joint maps.**

Proposition 2.4.1. *The map ψ_i is additive and quasimultiplicative with respect to the lattice operations + and $\cap$, namely:*

(1) $\psi_i(a) + \psi_i(b) = \psi_i(a+b)$,

(2) $\psi_i(a)\psi_i(b) = \psi_i((a+e_i)(b+x_jx_kx_l))$,

(3) $\psi_i(a)\psi_i(b) = \psi_i(a(b+e_i+x_jx_kx_l))$.

For the proof, see [St04, Prop. 4.4.1] □

We need the following corollary (atomic multiplicativity) from Proposition 2.4.1

Corollary 2.4.2. (a) *Suppose one of the following inclusions holds:*

$$\begin{array}{ll} \text{(i)} & e_i + e_je_ke_l \subseteq a, \\ \text{(ii)} & e_i + e_je_ke_l \subseteq b, \\ \text{(iii)} & e_i \subseteq a, \quad e_je_ke_l \subseteq b, \\ \text{(iv)} & e_i \subseteq b, \quad e_je_ke_l \subseteq a. \end{array}$$

Then the joint map ψ_i operates as a homomorphism on the elements a and b with respect to the lattice operations + and $\cap$, i.e.,

$$\psi_i(a) + \psi_i(b) = \psi_i(a+b), \quad \psi_i(a)\psi_i(b) = \psi_i(a)\psi_i(b).$$

(b) *The joint map ψ_i applied to the following atomic elements is the intersection preserving map, i.e., multiplicative with respect to the operation $\cap$:*

$$\psi_i(ba_n^{ij}) = \psi_i(b)\psi_i(a_n^{ij}) \text{ for every } b \subseteq D^4. \tag{2.17}$$

2.5. **The action of maps ψ_i and φ_i on the atomic elements.**

Proposition 2.5.1. *The joint maps ψ_i applied to the atomic elements a_n^{ij} satisfy the following relations*

(1) $\psi_i(a_n^{ij}) = \nu^0(e_ia_n^{kl})$,

(2) $\psi_j(a_n^{ij}) = \nu^0(e_j(e_k+e_l))$,

(3) $\psi_j(e_ia_n^{kl}) = \nu^0(e_ja_{n+1}^{kl})$.

For the proof, see [St04, Prop. 4.5.1]

2.6. **The fundamental property of the elementary maps.**

Proposition 2.6.1. *For $\{i,j,k,l\} = \{1,2,3,4\}$ the following relations hold*

$$\varphi_i\varphi_k\varphi_j + \varphi_i\varphi_l\varphi_j = 0, \tag{2.18}$$

$$\varphi_i^2 = 0. \tag{2.19}$$

Proof. For every vector $v \in X_0^1$, by definition of φ_i we have $(\varphi_i + \varphi_j + \varphi_k + \varphi_l)(v) = 0$, see eq. (2.10). In other words, $\varphi_i + \varphi_j + \varphi_k + \varphi_l = 0$. Therefore,

$$\varphi_i\varphi_k\varphi_j + \varphi_i\varphi_l\varphi_j = \varphi_i(\varphi_i + \varphi_j)\varphi_j = \varphi_i^2\varphi_j + \varphi_i\varphi_j^2.$$

So, it suffices to prove that $\varphi_i^2 = 0$. For every $z \subseteq D^4$, by Corollary 2.3.2, headings (3) and (1), we have

$$\begin{aligned} &\varphi_i^2\rho_{X^2}(z) \subseteq \varphi_i(\varphi_i\rho_{X^2}(I)) = \\ &\varphi_i(\rho_{X^+}(e_i(e_j+e_k+e_l))) \subseteq \varphi_i\rho_{X^+}(e_i) = 0. \quad \square \end{aligned}$$

Corollary 2.6.2. *The relation*

$$\varphi_i\varphi_k\varphi_j(B) = \varphi_i\varphi_l\varphi_j(B) \tag{2.20}$$

takes place for every subspace $B \subseteq X_0^2$*, where* $X^2 = X_0^2$ *is the representation space of* ρ_{X^2}*.*

Essentially, relations (2.18) and (2.20) are fundamental and motivate the construction of the admissible sequences satisfying the following relation:

$$ikj = ilj, \tag{2.21}$$

where indices i, j, k, l are all distinct, see Table 2.2 and Section 2.1.

2.7. **The** φ_i-**homomorphic elements.** By analogy with the modular lattice $D^{2,2,2}$ (see [St04, prop. Sect. 4.7]), we introduce now φ_i-homomorphic polynomials in D^4.

An element $a \subseteq D^4$ is said to be φ_i-*homomorphic*, if

$$\varphi_i\rho_{X^+}(ap) = \varphi_i\rho_{X^+}(a)\varphi_i\rho_{X^+}(p) \text{ for all } p \subseteq D^4. \tag{2.22}$$

An element $a \subseteq D^4$ is said to be $(\varphi_i, e_k)-$*homomorphic*, if

$$\varphi_i\rho_{X^+}(ap) = \varphi_i\rho_{X^+}(e_k a)\varphi_i\rho_{X^+}(p) \text{ for all } p \subseteq e_k. \tag{2.23}$$

Theorem 2.7.1. *1) The polynomials* a_n^{ij} *are* φ_i-*homomorphic.*

2) The polynomials a_n^{ij} *are* $(\varphi_j, e_k)-$*homomorphic for distinct indices* $\{i, j, k\}$*.*

For the proof, see [St04, Th. 1.7.1]. □

2.8. **The theorem on the classes of admissible elements.**

Theorem 2.8.1. *Let* $\alpha = i_n i_{n-1} \dots 1$ *be an admissible sequence for* D^4 *and* $i \neq i_n$*. Then* $i\alpha$ *is admissible and, for* $z_\alpha = e_\alpha$ *or* $f_{\alpha 0}$ *from Table 2.3, the following relation holds:*

$$\varphi_i\rho_{X^+}(z_\alpha) = \rho_X(z_{i\alpha}). \tag{2.24}$$

For the proof of the theorem on admissible elements in D^4, see Section B.2 in [St04].

The proof repeatedly uses the basic properties of the admissible sequences in D^4 considered in Section 2.8.1, Lemma 2.8.2.

2.8.1. *Basic properties of admissible elements in* D^4*.* We prove here a number of basic properties. of the atomic elements in D^4 used in the proof of the theorem on admissible elements (Theorem 2.8.1). In particular, in some cases the lower indices of polynomials a_s^{ij} entering in the admissible elements $f_{\alpha 0}$ can be transformed as in the following

Lemma 2.8.2. *1) Every polynomial* $f_{\alpha 0}$ *from Table 2.3 can be represented as an intersection of* e_α *and* P*. For every* $i \neq i_n$ *(see Section 1.4), we select* P *to be some* φ_i-*homomorphic polynomial.*

2) The lower indices of polynomials a_s^{ij} *entering in the admissible elements* $f_{\alpha 0}$ *can be equalized as follows:*

$$\begin{aligned} f_{\alpha 0} = & f_{(21)^t(41)^r(31)^s 0} = \\ & e_\alpha(e_2 a_{2t}^{34} + a_{2r+1}^{41} a_{2s-1}^{31}) = e_\alpha(a_{2t}^{43} + e_1 a_{2r}^{24} a_{2s}^{23}). \end{aligned} \tag{2.25}$$

The generic relation[3] *is the following:*

$$\begin{aligned} e_i(e_i a_t^{jl} + & a_{r+1}^{kj} a_{s-1}^{kl}) = \\ & e_i(a_t^{jl} + e_i a_{r+1}^{kj} a_{s-1}^{kl}) = e_i(a_t^{jl} + e_k a_r^{ij} a_s^{il}). \end{aligned} \tag{2.26}$$

[3]Throughout this lemma we suppose that $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

	Admissible sequence α	Admissible polynomial e_α	Admissible polynomial $f_{\alpha 0}$
$F21$	$(21)^t(41)^r(31)^s =$ $(21)^t(31)^s(41)^r$	$e_2a^{31}_{2s}a^{41}_{2r}a^{34}_{2t-1}$	$e_\alpha(e_2a^{34}_{2t} + a^{41}_{2r+1}a^{31}_{2s-1}) =$ $e_\alpha(a^{43}_{2t} + e_1a^{24}_{2r}a^{23}_{2s})$
$F31$	$(31)^s(41)^r(21)^t =$ $(31)^s(21)^t(41)^r$	$e_3a^{21}_{2t}a^{41}_{2r}a^{24}_{2s-1}$	$e_\alpha(e_3a^{42}_{2s} + a^{41}_{2r+1}a^{21}_{2t-1}) =$ $e_\alpha(a^{42}_{2s} + e_1a^{34}_{2r}a^{32}_{2t})$
$F41$	$(41)^r(31)^s(21)^t =$ $(41)^r(21)^t(31)^s$	$e_4a^{21}_{2t}a^{31}_{2s}a^{32}_{2r-1}$	$e_\alpha(e_4a^{32}_{2r} + a^{31}_{2s+1}a^{21}_{2t-1}) =$ $e_\alpha(a^{32}_{2r} + e_1a^{43}_{2s}a^{42}_{2t})$
$G11$	$1(41)^r(31)^s(21)^t =$ $1(31)^s(41)^r(21)^t =$ $1(21)^t(31)^s(41)^r$	$e_1a^{24}_{2s}a^{34}_{2t}a^{32}_{2r}$	$e_\alpha(e_1a^{32}_{2r+1} + a^{24}_{2s+1}a^{34}_{2t-1}) =$ $e_\alpha(a^{32}_{2r+1} + e_4a^{21}_{2s}a^{31}_{2t})$
$G21$	$2(41)^r(31)^s(21)^t =$ $2(31)^{s+1}(41)^{r-1}(21)^t$	$e_2a^{34}_{2t}a^{31}_{2s+1}a^{14}_{2r-1} =$ $e_2a^{34}_{2t}a^{31}_{2s-1}a^{14}_{2r+1}$	$e_\alpha(e_2a^{14}_{2r} + a^{31}_{2s+2}a^{34}_{2t-1}) =$ $e_\alpha(a^{14}_{2r} + e_3a^{21}_{2s+1}a^{24}_{2t})$
$G31$	$3(41)^r(21)^t(31)^s =$ $3(21)^{t+1}(41)^{r-1}(31)^s$	$e_3a^{24}_{2s}a^{21}_{2t+1}a^{14}_{2r-1} =$ $e_3a^{24}_{2s}a^{21}_{2t-1}a^{14}_{2r+1}$	$e_\alpha(e_3a^{14}_{2r} + a^{21}_{2t+2}a^{24}_{2s-1}) =$ $e_\alpha(a^{14}_{2r} + e_2a^{31}_{2t+1}a^{34}_{2s})$
$G41$	$4(21)^t(31)^s(41)^r =$ $4(31)^{s+1}(21)^{t-1}(41)^r$	$e_4a^{32}_{2r}a^{31}_{2s+1}a^{12}_{2t-1} =$ $e_4a^{32}_{2r}a^{31}_{2s-1}a^{12}_{2t+1}$	$e_\alpha(e_4a^{12}_{2t} + a^{31}_{2s}a^{32}_{2r+1}) =$ $e_\alpha(a^{12}_{2t} + e_3a^{41}_{2s+1}a^{42}_{2r})$
$H11$	$(14)^r(31)^s(21)^t =$ $(14)^r(21)^{t+1}(31)^{s-1} =$ $(13)^s(41)^r(21)^t =$ $(13)^s(21)^{t+1}(41)^{r-1}$ $(12)^{t+1}(41)^r(31)^{s-1} =$ $(12)^{t+1}(31)^{s-1}(41)^r$	$e_1a^{23}_{2r+1}a^{24}_{2s-1}a^{34}_{2t-1} =$ $e_1a^{23}_{2r-1}a^{24}_{2s-1}a^{34}_{2t+1} =$ $e_1a^{23}_{2r-1}a^{24}_{2s+1}a^{34}_{2t-1}$	$e_\alpha(e_1a^{23}_{2r} + a^{24}_{2s}a^{34}_{2t}) =$ $e_\alpha(e_1a^{24}_{2s} + a^{34}_{2t}a^{23}_{2r}) =$ $e_\alpha(e_1a^{34}_{2t} + a^{23}_{2r}a^{24}_{2s})$

TABLE 2.3. Admissible polynomials in the modular lattice D^4

Notes to Table:

1) For more details about admissible sequences, see Proposition 2.1.3 and Table 2.1.

2) For relations given in two last columns (definitions of admissible polynomials e_α and $f_{\alpha 0}$), see Lemma 2.8.2.

3) In each line, each low index should be non-negative. For example, for Line $F21$, we have: $s \geq 0, r \geq 0, t \geq 1$; for Line $G21$, we have: $s \geq 0, r \geq 0, t \geq 0$.

3) The substitution

$$\begin{cases} r \mapsto r-2, \\ s \mapsto s+2 \end{cases} \tag{2.27}$$

does not change the polynomial $e_i(e_ia^{jl}_t + a^{kj}_{r+1}a^{kl}_{s-1})$, *namely:*

$$e_i(e_ia^{jl}_t + a^{kj}_{r+1}a^{kl}_{s-1}) = e_i(e_ia^{jl}_t + a^{kj}_{r-1}a^{kl}_{s+1}). \tag{2.28}$$

N	Equivalent form of $f_{\alpha 0}$	Forms obtained by $e_j a^{kl}_n = e_j a^{lk}_n$
1	$e_\alpha(e_2 a^{34}_{2t} + a^{41}_{2r+1} a^{31}_{2s-1})$	$e_\alpha(e_2 a^{43}_{2t} + a^{41}_{2r+1} a^{31}_{2s-1})$
2	$e_\alpha(a^{34}_{2t} + e_2 a^{41}_{2r+1} a^{31}_{2s-1})$	$e_\alpha(a^{43}_{2t} + e_2 a^{41}_{2r+1} a^{31}_{2s-1})$
3	$e_\alpha(a^{34}_{2t} + e_2 a^{14}_{2r+1} a^{13}_{2s-1})$	$e_\alpha(a^{43}_{2t} + e_2 a^{14}_{2r+1} a^{13}_{2s-1})$
4	$e_\alpha(e_2 a^{34}_{2t} + a^{14}_{2r+1} a^{13}_{2s-1})$	$e_\alpha(e_2 a^{43}_{2t} + a^{14}_{2r+1} a^{13}_{2s-1})$
5	$e_\alpha(a^{14}_{2r+1} + e_2 a^{34}_{2t} a^{13}_{2s-1})$	$e_\alpha(a^{14}_{2r+1} + e_2 a^{43}_{2t} a^{13}_{2s-1})$
6	$e_\alpha(e_2 a^{14}_{2r+1} + a^{34}_{2t} a^{13}_{2s-1})$	$e_\alpha(e_2 a^{14}_{2r+1} + a^{43}_{2t} a^{13}_{2s-1})$
7	$e_\alpha(a^{34}_{2t} + e_2 a^{14}_{2r-1} a^{13}_{2s+1})$	$e_\alpha(a^{43}_{2t} + e_2 a^{14}_{2r-1} a^{13}_{2s+1})$
8	$e_\alpha(a^{34}_{2t} a^{14}_{2r-1} + e_2 a^{13}_{2s+1})$	$e_\alpha(a^{43}_{2t} a^{14}_{2r-1} + e_2 a^{13}_{2s+1})$
9	$e_\alpha(a^{13}_{2s+1} + e_2 a^{34}_{2t} a^{14}_{2r-1})$	$e_\alpha(a^{13}_{2s+1} + e_2 a^{43}_{2t} a^{14}_{2r-1})$

TABLE 2.4. Different equivalent forms of the element $f_{\alpha 0} = f_{(21)^t (41)^r (31)^s 0}$

4) The substitution (2.27) does not change the polynomial $e_i a^{kj}_s a^{kl}_r$:

$$e_i a^{kj}_{r+1} a^{kl}_{s-1} = e_i a^{kj}_{r-1} a^{kl}_{s+1}. \tag{2.29}$$

For the proof of this lemma see [St04, Lemma 4.8.2]. □

2.8.2. *Coincidence with the Gelfand-Ponomarev polynomials in D^4/θ.* Let $e_\alpha, f_{\alpha 0}$ be admissible elements constructed in this work and $\tilde{e}_\alpha, \tilde{f}_{\alpha 0}$ be the admissible elements constructed by Gelfand and Ponomarev [GP74]. We will prove that for the admissible sequences of the small length, the coincidence of e_α with $\tilde{e}_\alpha$ (resp. $f_{\alpha 0}$ with $\tilde{f}_{\alpha 0}$) takes place in D^4 (not only in D^4/θ). Since Theorem 2.8.1 takes place for both $e_\alpha, f_{\alpha 0}$ and for $\tilde{e}_\alpha, \tilde{f}_{\alpha 0}$ [GP74, Th.7.2, Th.7.3] we have

Proposition 2.8.3. *The elements e_α, (resp. $f_{\alpha 0}$) and $\tilde{e}_\alpha$ (resp. $\tilde{f}_{\alpha 0}$) coincide in D^4/θ.*

Recall definitions of $\tilde{e}_\alpha$ and $\tilde{f}_{\alpha 0}$ from [GP74].

<u>The definition of $\tilde{e}_\alpha$, [GP74, p.6]</u>.

$$\tilde{e}_{i_n i_{n-1} \dots i_2 i_1} = \tilde{e}_{i_n} \sum_{\beta \in \Gamma_e(\alpha)} \tilde{e}_\beta, \tag{2.30}$$

where

$$\begin{aligned} \Gamma_e(\alpha) = \{\beta = (k_{n-1}, \dots, k_2, k_1) \mid k_{n-1} \notin \{i_n, i_{n-1}\}, \dots, k_1 \notin \{i_2, i_1\}, \text{ and} \\ k_1 \neq k_2, \dots, k_{n-2} \neq k_{n-1}\}. \end{aligned} \tag{2.31}$$

<u>The definition of $\tilde{f}_{\alpha 0}$, [GP74, p.53]</u>.

$$\tilde{f}_{i_n i_{n-1} \dots i_2 i_1 0} = \tilde{e}_{i_n} \sum_{\beta \in \Gamma_f(\alpha)} \tilde{e}_\beta, \tag{2.32}$$

where

$$\begin{aligned} \Gamma_f(\alpha) = \{\beta = (k_n, \dots, k_2, k_1) \mid k_n \notin \{i_n, i_{n-1}\}, \dots, k_2 \notin \{i_2, i_1\}, k_1 \notin \{i_1\} \text{ and} \\ k_1 \neq k_2, \dots, k_{n-2} \neq k_{n-1}\}. \end{aligned} \tag{2.33}$$

Proposition 2.8.4 (The elements $\tilde{e}_\alpha$)**.** *Consider elements $\tilde{e}_\alpha$ for $\alpha = 21, 121, 321, 2341$ (see Section 1.6). The relation*

$$e_\alpha = \tilde{e}_\alpha$$

takes place in D^4.

Proof. For $n = 2$: $\alpha = 21$. We have

$$\tilde{e}_{21} = e_2 \sum_{j \neq 1,2} e_j = e_2(e_3 + e_4). \tag{2.34}$$

According to Section 1.6, we see that $e_{21} = \tilde{e}_{21}$.

For $n = 3$: 1) $\alpha = 121$,

$$\Gamma_e(\alpha) = \{(k_2 k_1) \mid k_2 \in \{3,4\}, k_1 \in \{3,4\}, k_1 \neq k_2\},$$
$$\tilde{e}_{121} = e_1 \sum_{\beta \in \Gamma_e(\alpha)} e_\beta = e_1(e_{34} + e_{43}) = e_1(e_3(e_1 + e_2) + e_4(e_1 + e_2)) = e_1 a_2^{34}. \tag{2.35}$$

By Section 1.6 we have $e_{121} = \tilde{e}_{121}$.

2) $\alpha = 321 = 341$. We have

$$\begin{aligned}
\Gamma_e(\alpha) =& \{(k_2 k_1) \mid k_2 \in \{3,2\}, k_1 \in \{2,1\}, k_1 \neq k_2\} = \{14, 13, 43\}.\\
\tilde{e}_{321} =& e_1 \sum_{\beta \in \Gamma_e(\alpha)} e_\beta = e_3(e_{14} + e_{13} + e_{43}) = \\
& e_3(e_1(e_2 + e_4) + e_1(e_2 + e_3) + e_4(e_1 + e_2)) = \\
& e_3((e_1 + e_2)(e_2 + e_4)(e_1 + e_4) + e_1(e_2 + e_3)) = \\
& e_3((e_1 + e_2)(e_1 + e_4)(e_2 + e_4 + e_1(e_2 + e_3)) = \\
& e_3((e_1 + e_2)(e_1 + e_4)(e_4 + (e_1 + e_2)(e_2 + e_3)) = \\
& e_3((e_1 + e_2)(e_1 + e_4)(e_4(e_2 + e_3) + (e_1 + e_2)).
\end{aligned}$$

Since $e_1 + e_2 \subseteq e_4(e_2 + e_3) + e_1 + e_2$, we have

$$\tilde{e}_{321} = e_3(e_{14} + e_{13} + e_{43}) = e_3(e_1 + e_2)(e_1 + e_4). \tag{2.36}$$

Since $\tilde{e}_{321}$ is symmetric with respect to transposition $2 \leftrightarrow 4$, we have

$$\tilde{e}_{321} = \tilde{e}_{341} = e_3(e_{14} + e_{13} + e_{43}) = e_3(e_{12} + e_{13} + e_{23}) = e_3(e_1 + e_2)(e_1 + e_4). \tag{2.37}$$

By Section 1.6 we have $e_{321} = \tilde{e}_{321}$.

For $n = 4$: $\alpha = 2341 = 2321 = 2141$. We have

$$\begin{aligned}
\Gamma_e(\alpha) =& \{(k_3 k_2 k_1) \mid k_3 \in \{1,4\}, k_2 \in \{1,2\}, k_1 \in \{2,3\}, \quad k_1 \neq k_2, k_2 \neq k_3\} = \\
& \{(123), (412), (413) = (423)\},
\end{aligned} \tag{2.38}$$

and

$$\begin{aligned}
\tilde{e}_{2341} =& e_2 \sum_{\beta \in \Gamma_e(\alpha)} e_\beta = e_2(e_{123} + e_{412} + e_{413}) = \\
& e_2(e_1(e_2 + e_3)(e_4 + e_3) + e_4(e_1 + e_2)(e_3 + e_2) + e_4(e_1 + e_3)(e_2 + e_3)) = \\
& e_2(e_4 + e_3)(e_1(e_2 + e_3) + e_4(e_1 + e_2)(e_3 + e_2) + e_4(e_1 + e_3)(e_3 + e_2)) = \\
& e_2(e_4 + e_3)(e_1(e_2 + e_3) + e_4 + e_4(e_1 + e_3)(e_1 + e_2)(e_3 + e_2)) = \\
& e_2(e_4 + e_3)(e_1(e_2 + e_3) + e_4).
\end{aligned} \tag{2.39}$$

By Section 1.6 we have $e_{2341} = \tilde{e}_{2341}$. The proposition is proved. $\square$

Proposition 2.8.5 (The elements $\tilde{f}_{\alpha 0}$)**.** *Consider elements $\tilde{f}_{\alpha 0}$ for $\alpha = 21, 121, 321$ (see Section 1.6). The following relation*

$$f_{\alpha 0} = \tilde{f}_{\alpha 0}$$

takes place in D^4.

Proof. For $n = 2$: $\alpha = 21$. We have

$$\begin{aligned}
&\Gamma_f(\alpha) = \{(k_2 k_1) \mid k_2 \in \{3,4\}, k_1 \in \{2,3,4\}, k_1 \neq k_2\},\\
\tilde{f}_{210} =& e_2 \sum_{\beta \in \Gamma_f(\alpha)} e_\beta = e_2(e_{32} + e_{34} + e_{42} + e_{43}) = \\
& e_2(e_3(e_1 + e_4) + e_3(e_2 + e_1) + e_4(e_1 + e_3) + e_4(e_1 + e_2)) = \\
& e_2((e_3 + e_4)(e_1 + e_4)(e_1 + e_3) + e_3(e_1 + e_2) + e_4(e_1 + e_2)) = \\
& e_2(e_3 + e_4)((e_1 + e_4)(e_1 + e_3) + e_3(e_1 + e_2) + e_4(e_1 + e_2)) = \\
& e_2(e_3 + e_4)(e_1 + e_4(e_1 + e_3) + e_3(e_1 + e_2) + e_4(e_1 + e_2)) = \\
& e_2(e_3 + e_4)(e_4(e_1 + e_3) + (e_1 + e_3)(e_1 + e_2) + e_4(e_1 + e_2)) = \\
& e_2(e_3 + e_4)(e_4(e_1 + e_3)(e_1 + e_2) + (e_1 + e_3)(e_1 + e_2) + e_4) = \\
& e_2(e_3 + e_4)(e_4 + (e_1 + e_3)(e_1 + e_2)) = \\
& e_2(e_3 + e_4)(e_4 + e_1 + e_3(e_1 + e_2)).
\end{aligned} \tag{2.40}$$

By Section 1.6 we have $f_{210} = \tilde{f}_{210}$.

For $n = 3$: 1) $\alpha = 121$. For this case, we have

$$\Gamma_f(\alpha) = \{(k_3 k_2 k_1) \mid k_3 \in \{3,4\}, k_2 \in \{3,4\}, k_1 \in \{2,3,4\}, k_1 \neq k_2, k_2 \neq k_3\}, \tag{2.41}$$

and

$$\begin{aligned}
\tilde{f}_{1210} =& e_1 \sum_{\beta \in \Gamma_f(\alpha)} e_\beta = e_1(e_{342} + e_{343} + e_{432} + e_{434}) = \\
& e_1[e_3(e_1 + e_2)(e_4 + e_2) + e_3(e_1(e_4 + e_3) + e_2(e_4 + e_3)) + \\
& \quad e_4(e_1 + e_2)(e_3 + e_2) + e_4(e_1(e_4 + e_3) + e_2(e_4 + e_3))] = \\
& e_1[e_3(e_1 + e_2)(e_4 + e_2 + e_3(e_1 + e_2(e_4 + e_3))) + \\
& \quad e_4(e_1 + e_2)(e_3 + e_2 + e_4(e_1 + e_2(e_4 + e_3)))] = \\
& e_1[e_3(e_1 + e_2)(e_4 + e_2 + e_1(e_3 + e_2(e_4 + e_3))) + \\
& \quad e_4(e_1 + e_2)(e_3 + e_2 + e_1(e_4 + e_2(e_4 + e_3)))] = \\
& e_1[(e_4 + e_2 + e_1(e_3 + e_2)(e_4 + e_3))] \\
& \quad [(e_3 + e_2 + e_1(e_4 + e_2)(e_4 + e_3))] \\
& \quad [e_3(e_1 + e_2) + e_4(e_1 + e_2)] = \\
& e_1[(e_4 + e_2)(e_4 + e_3) + e_1(e_3 + e_2))] \\
& \quad [(e_3 + e_2 + e_1(e_4 + e_2)(e_4 + e_3))] \\
& \quad [e_3(e_1 + e_2) + e_4(e_1 + e_2)] = \\
& e_1[(e_4 + e_2)(e_4 + e_3) + e_1(e_3 + e_2))] \\
& \quad [e_1(e_3 + e_2) + (e_4 + e_2)(e_4 + e_3))] \\
& \quad [e_3(e_1 + e_2) + e_4(e_1 + e_2)].
\end{aligned} \tag{2.42}$$

Since the first two intersection polynomials in the last expression of $\tilde{f}_{1210}$ coincide with

$$(e_4+e_2)(e_4+e_3)+e_1(e_3+e_2)),$$

we have

$$\tilde{f}_{1210}=e_1((e_4+e_2)(e_4+e_3)+e_1(e_3+e_2)))(e_3+e_4(e_1+e_2)). \tag{2.43}$$

By Section 1.6 we have $f_{1210}=\tilde{f}_{1210}$.

2) $\alpha=321$. Here we have

$$\begin{aligned}&\Gamma_f(\alpha)=\{(k_3k_2k_1)\mid k_3\in\{1,4\},k_2\in\{3,4\},k_1\in\{2,3,4\},k_1\neq k_2,k_2\neq k_3\}=\\&\{(132)=(142),(134),(143,(432),(434)\}\end{aligned} \tag{2.44}$$

and

$$\begin{aligned}\tilde{f}_{3210}=&e_1\sum_{\beta\in\Gamma_f(\alpha)}e_\beta=e_3(e_{132}+e_{134}+e_{143}+e_{432}+e_{434})=\\&e_3[e_1(e_4+e_2)(e_3+e_2)+e_1(e_4+e_2)(e_3+e_4)+\\&\quad e_1(e_3+e_2)(e_3+e_4)+e_4(e_3+e_2)(e_1+e_2)+\\&\quad e_4(e_1+e_2(e_3+e_4))].\end{aligned} \tag{2.45}$$

Since

$$\begin{aligned}&e_1(e_4+e_2)(e_3+e_4)+e_4(e_1+e_2(e_3+e_4))=\\&(e_3+e_4)(e_4+e_2)(e_1+e_4(e_1+e_2(e_3+e_4))=\\&(e_3+e_4)(e_4+e_2)(e_1+e_4)(e_1+e_2(e_3+e_4)),\end{aligned} \tag{2.46}$$

by (2.45) we have

$$\begin{aligned}&\tilde{f}_{3210}=\\&\quad e_3[e_1(e_4+e_2)(e_3+e_2)+\\&\qquad(e_3+e_4)(e_4+e_2)(e_1+e_4)(e_1+e_2(e_3+e_4))+\\&\qquad e_1(e_3+e_2)(e_3+e_4)+e_4(e_3+e_2)(e_1+e_2)]=\\&\quad e_3(e_1+e_2)(e_1+e_4)[e_1(e_4+e_2)(e_3+e_2)+\\&\qquad(e_3+e_4)(e_4+e_2)(e_1+e_2(e_3+e_4))+\\&\qquad e_1(e_3+e_2)(e_3+e_4)+e_4(e_3+e_2)]=\\&\quad e_3(e_1+e_2)(e_1+e_4)[e_1(e_4+e_2)(e_3+e_2)(e_3+e_4)+\\&\qquad(e_3+e_4)(e_4+e_2)(e_1+e_2(e_3+e_4))+\\&\qquad e_1(e_3+e_2)+e_4(e_3+e_2)].\end{aligned} \tag{2.47}$$

Since

$$e_1(e_4+e_2)(e_3+e_2)(e_3+e_4)\subseteq e_1(e_3+e_2),$$

by (2.47) we have

$$\begin{aligned}
&\tilde{f}_{3210} = \\
&\quad e_3(e_1+e_2)(e_1+e_4)[(e_3+e_4)(e_4+e_2)(e_1+e_2(e_3+e_4))+ \\
&\qquad e_1(e_3+e_2)+e_4(e_3+e_2)] = \\
&\quad e_3(e_1+e_2)(e_1+e_4)[e_1(e_3+e_4)(e_4+e_2)+e_2(e_3+e_4))+ \\
&\qquad e_1(e_3+e_2)+e_4(e_3+e_2)] = \\
&\quad e_3(e_1+e_2)(e_1+e_4)[e_1(e_3+e_4)(e_4+e_2)(e_3+e_2)+e_2(e_3+e_4))+ \\
&\qquad e_1+e_4(e_3+e_2)].
\end{aligned} \tag{2.48}$$

Since

$$e_1(e_3+e_4)(e_4+e_2)(e_3+e_2) \subseteq e_1,$$

by (2.48) we have

$$\begin{aligned}
&\tilde{f}_{3210} = \\
&\quad e_3(e_1+e_2)(e_1+e_4)(e_2(e_3+e_4))+e_1+e_4(e_3+e_2)) = \\
&\quad e_3(e_1+e_2)(e_1+e_4)(e_1+(e_2+e_4)(e_3+e_2)(e_3+e_4)).
\end{aligned} \tag{2.49}$$

By Section 1.6 we have $f_{3210} = \tilde{f}_{3210}$. The proposition is proved. $\square$

Conjecture 2.8.6. For every admissible sequence α, the elements e_α (resp. $f_{\alpha 0}$) and $\tilde{e}_\alpha$ (resp. $\tilde{f}_{\alpha 0}$) coincide in D^4 (see Proposition 2.8.3).

In Propositions 2.8.4 and 2.8.5, this conjecture was proven for small values of lengths of the admissible sequence α.

3. Admissible elements in D^4 and Herrmann's polynomials

In this section we consider Herrmann's endomorphisms γ_{ik} $(i, k = 1, 2, 3)$ and polynomials s_n, t_n, $p_{i,n}$ $(i = 1, 2, 3, 4)$ being perfect elements, [H82]. Endomorphisms γ_{ik} are important in construction of the perfect elements s_n, t_n, $p_{i,n}$, the perfect elements constitute 16-element Boolean cube, coinciding modulo linear equivalence with the Gelfand-Ponomarev boolean cube $B^+(n)$, see Theorem 3.4.4 due to Herrmann, [H82].

We show that endomorphisms γ_{ik} are also closely connected with admissible elements.

First of all, the endomorphism γ_{ik} acts on the admissible element $e_{\alpha k}$ such that

$$\gamma_{ik}(e_{\alpha k}) = e_{\alpha k i},$$

see Theorem 3.3.4. Thus, endomorphism γ_{ik} acts also on the admissible sequence. We see some similarity between the action of endomorphism γ_{ik} and the action of elementary map of Gelfand-Ponomarev ϕ_i. The endomorphism γ_{ik} and the elementary map ϕ_i act in a sense in opposite directions, namely the endomorphism γ_{ik} adds the index to the start of the admissible sequence, and the elementary map ϕ_i adds the index to the end[4].

Further, the endomorphisms γ_{ik} commute and we can consider a sequence of these endomorphism. Admissible elements $e_1 a_t^{34} a_s^{24} a_r^{32}$ from Table 3.1 are obtained by means of Herrmann's endomorphisms as follows:

$$\gamma_{12}^t \gamma_{13}^s \gamma_{14}^r (e_1) = e_1 a_t^{34} a_s^{24} a_r^{32},$$

see Theorem 3.3.6.

[4]Recall, that in the admissible sequence $i_n i_{n-1} \dots i_2 i_1$ the index i_1 is the start and i_n is the end.

At last, we will see how Herrmann's polynomials s_n, t_n, $p_{i,n}$ are expressed by means of the cumulative elements $e_i(n)$, $f_i(n)$:

$$
\begin{aligned}
s_n &= \sum_{i=1,2,3,4} e_i(n),\\
t_n &= f_0(n+1),\\
p_{i,n} &= e_i(n) + f_0(n+1),
\end{aligned}
$$

see Theorem 3.4.3 and Table 1.1.

3.1. **An unified formula of admissible elements.** It turned out, that the admissible elements e_α and $e_{\alpha 0}$ described by Table 2.3 can be written by an unified formula. This unified formula represented in Table 3.1 and Proposition 3.1.1.

A main difference between Table 2.3 and Table 3.1 is in following: Table 2.3 describes admissible elements with admissible sequence α starting at generator e_1, and Table 3.1 describes admissible elements with admissible sequence α ending at generator e_1. Recall, that type Fij (resp. Gij, $H11$) denotes the admissible sequence starting at j and ending at i, see Remark 2.1.2.

As we will see in the next sections, the unified formula represented in this section in Table 3.1 and in Proposition 3.1.1 is a basis of the construction of Herrmann's polynomials, [H82], [H84].

	Admissible polynomial e_α	Admissible polynomial $f_{\alpha 0}$	Signature of polynomial e_α
$F12$	$e_1 a^{32}_{2r} a^{42}_{2s} a^{34}_{2t-1}$	$e_\alpha(e_1 a^{34}_{2t} + a^{42}_{2s+1} a^{32}_{2r-1})$	0 0 1
$F13$	$e_1 a^{32}_{2r} a^{42}_{2s-1} a^{34}_{2t}$	$e_\alpha(e_1 a^{34}_{2t+1} + a^{42}_{2s} a^{32}_{2r-1})$	0 1 0
$F14$	$e_1 a^{32}_{2r-1} a^{42}_{2s} a^{34}_{2t}$	$e_\alpha(e_1 a^{34}_{2t+1} + a^{42}_{2s+1} a^{32}_{2r-2})$	1 0 0
$G11$	$e_1 a^{32}_{2r} a^{42}_{2s} a^{34}_{2t}$	$e_\alpha(e_1 a^{34}_{2t+1} + a^{42}_{2s+1} a^{32}_{2r-1})$	0 0 0
$G12$	$e_1 a^{32}_{2r+1} a^{42}_{2s-1} a^{34}_{2t}$	$e_\alpha(e_1 a^{34}_{2t+1} + a^{42}_{2s} a^{32}_{2r})$	1 1 0
$G13$	$e_1 a^{32}_{2r-1} a^{42}_{2s} a^{34}_{2t+1}$	$e_\alpha(e_1 a^{34}_{2t} + a^{42}_{2s+1} a^{32}_{2r-2})$	1 0 1
$G14$	$e_1 a^{32}_{2r} a^{42}_{2s-1} a^{34}_{2t+1}$	$e_\alpha(e_1 a^{34}_{2t+2} + a^{42}_{2s} a^{32}_{2r-1})$	0 1 1
$H11$	$e_1 a^{32}_{2r-1} a^{42}_{2s+1} a^{34}_{2t-1}$	$e_\alpha(e_1 a^{34}_{2t} + a^{42}_{2s+2} a^{32}_{2r-2})$	1 1 1

TABLE 3.1. Signatures of admissible elements e_α and $f_{\alpha 0}$ ending at 1

Proposition 3.1.1. (a) *The set of all admissible polynomials e_α (described by Table 2.3) with admissible sequences α ending at* 1 *coincides with the set of polynomials*

$$
e_1 a^{32}_r a^{24}_s a^{34}_t, \qquad \text{where } r, s, t = 0, 1, 2, \dots, \tag{3.1}
$$

see Table 3.1.

Similarly, the set of all admissible polynomials e_α with admissible sequences α ending at $i \in \{2,3,4\}$ coincides with the set of polynomials

$$e_i a_r^{jl} a_s^{lk} a_t^{kj}, \qquad \textit{where } r,s,t = 0,1,2,\dots, \\ \textit{and } \{i,j,k,l\} = \{1,2,3,4\}. \tag{3.2}$$

(b) *The set of all admissible polynomials $f_{\alpha 0}$, (described by Table 2.3) with admissible sequences α ending at* 1 *coincides with the set of polynomials*

$$e_\alpha(e_1 a_{t+1}^{34} + a_{s+1}^{42} a_{r-1}^{32}) = e_1 a_t^{34} a_s^{42} a_r^{32}(e_1 a_{t+1}^{34} + a_{s+1}^{42} a_{r-1}^{32}), \\ \textit{where } r,s,t = 0,1,2,\dots, \tag{3.3}$$

see Table 3.1.

Similarly, the set of all admissible polynomials $f_{\alpha 0}$ with admissible sequences α ending at $i \in \{2,3,4\}$ coincides with the set of polynomials

$$e_\alpha(e_i a_t^{jk} + a_s^{kl} a_r^{jl}) = e_i a_t^{jk} a_s^{kl} a_r^{lj}(e_i a_{t+1}^{jk} + a_{s+1}^{kl} a_{r-1}^{jl}), \\ \textit{where } r,s,t = 0,1,2,\dots, \textit{ and } \{i,j,k,l\} = \{1,2,3,4\}. \tag{3.4}$$

Proof. For any atomic element a_i^{pq}, set the range 1 if i is odd and 0 if i is even. For the admissible polynomials e_α from Table 3.1, the set of the corresponding ranges we call the *signature*. For example, for e_α of the type $F12$, the signature is $(0,0,1)$, see Table 3.1. There are 8 possible signatures. Since, all possible signatures appeared in Table 3.1, we can get all possible combination of indices (r,s,t) in (3.1), (3.3). □

Remark 3.1.2. For the admissible polynomials e_α the sum of low indices of atomic elements a_r^{kl} is by 1 less the length of the admissible sequence α, for the admissible polynomials $f_{\alpha 0}$ the sum of low indices of atomic elements a_r^{kl} (contained in the parantheses) is equal to the length of the admissible sequence α, see Table 2.3.

3.2. **Inverse cumulative elements in D^4.** In addition to cumulative elements, we introduce now *inverse cumulative elements.* A difference between cumulative elements and inverse cumulative elements is in following: the cumulative elements accumulate all admissible elements of the given length <u>starting</u> at some generator e_i, and inverse cumulative elements accumulate all admissible elements of the given length <u>ending</u> at some generator e_i.

The cumulative elements $e_i(n)$, where $i = 1,2,3,4$, are defined in (1.16). Proposition 3.1.1 and Remark 3.1.2 motivate the following definition of inverse cumulative polynomials $e_i^\vee(n)$, where $i = 1,2,3,4$, and $f_0^\vee(n)$ as follows:

$$\begin{aligned} e_1^\vee(n) &= \sum_{r+s+t=n-1} e_1 a_r^{32} a_s^{24} a_t^{34}, & e_2^\vee(n) &= \sum_{r+s+t=n-1} e_2 a_r^{31} a_s^{14} a_t^{34}, \\ e_3^\vee(n) &= \sum_{r+s+t=n-1} e_3 a_r^{12} a_s^{24} a_t^{14}, & e_4^\vee(n) &= \sum_{r+s+t=n-1} e_4 a_r^{12} a_s^{23} a_t^{13}, \end{aligned} \tag{3.5}$$

and

$$f_0^\vee(n) = \sum f_{1i_{n-1}\dots i_2 0} + \sum f_{2i_{n-1}\dots i_2 0} + \sum f_{3i_{n-1}\dots i_2 0} + \sum f_{4i_{n-1}\dots i_2 0} =$$

$$\sum_{r+s+t=n} e_\alpha(e_1 a_t^{34} + a_s^{24} a_r^{32}) + \sum_{r+s+t=n} e_\alpha(e_2 a_t^{34} + a_s^{14} a_r^{31}) + \tag{3.6}$$

$$\sum_{r+s+t=n} e_\alpha(e_3 a_t^{24} + a_s^{14} a_r^{21}) + \sum_{r+s+t=n} e_\alpha(e_4 a_t^{23} + a_s^{13} a_r^{21}).$$

Proposition 3.2.1. (a) *The sum of all cumulative elements $e_i(n)$ of the given length n and the sum of all inverse cumulative elements $e_i^\vee(n)$ coincide, i.e.,*

$$e_1(n) + e_2(n) + e_3(n) + e_4(n) = e_1^\vee(n) + e_2^\vee(n) + e_3^\vee(n) + e_4^\vee(n). \tag{3.7}$$

(b) *The cumulative element $f_0(n)$ coincides with inverse cumulative element $f_0^\vee(n)$:*

$$f_0(n) = f_0^\vee(n). \tag{3.8}$$

(c) *The number of elements in every sum (3.5) is $\frac{1}{2}(n+1)(n+2)$*[5].

Proof. (a), (b) are true since sums from the both sides consist of all admissible elements of the given length n.

(c) We just need to found the number of solutions of the equation

$$r + s + t = n. \tag{3.9}$$

These solutions are points with integer *barycentric coordinates*[6] in the triangle depicted on Fig. 3.1. Let (r, s, t) be coordinates of any point of this triangle. Any move along one of edges does not change the coordinate sum $r + s + t$, and this sum is equal to n. $\square$

3.3. **Herrmann's endomorphisms and admissible elements.** Herrmann introduced in [H82, p.361, p.367], [H84, p.229] polynomials q_{ij} and associated endomorphisms γ_{ij} of D^4 playing the central role in his study of the modular lattice D^4, in particular, in his construction of perfect polynomials.

For $\{i, j, k, l\} = \{1, 2, 3, 4\}$, define

$$q_{ij} = q_{ji} = q_{kl} = q_{lk} = (e_i + e_j)(e_k + e_l). \tag{3.10}$$

In our denotations,

$$q_{ij} = a_1^{ij} a_1^{kl}, \text{ where } \{i, j, k, l\} = \{1, 2, 3, 4\}. \tag{3.11}$$

The endomorphism γ_{ij} of D^4 is denoted as follows:

$$1 \mapsto q_{ij}, \quad 0 \mapsto 0, \quad e_k \mapsto e_k q_{ij}. \tag{3.12}$$

For every polynomial $f(e_1, e_2, e_3, e_4)$, we have

$$\gamma_{ij} f(e_1, e_2, e_3, e_4) = f(e_1 q_{ij}, e_2 q_{ij}, e_3 q_{ij}, e_4 q_{ij}).$$

Essentially, by (3.10) among endomorphisms γ_{ij}, there are only 3 different:

$$\gamma_{12}, \quad \gamma_{13}, \quad \gamma_{14}.$$

[5]Compare with Remark 2.1.4.

[6]For details concerning barycentric coordinates, see, e.g., H. S. M. Coxeter's book, [Cox89, Section 13.7], or A. Bogomolny's site [Bm96].

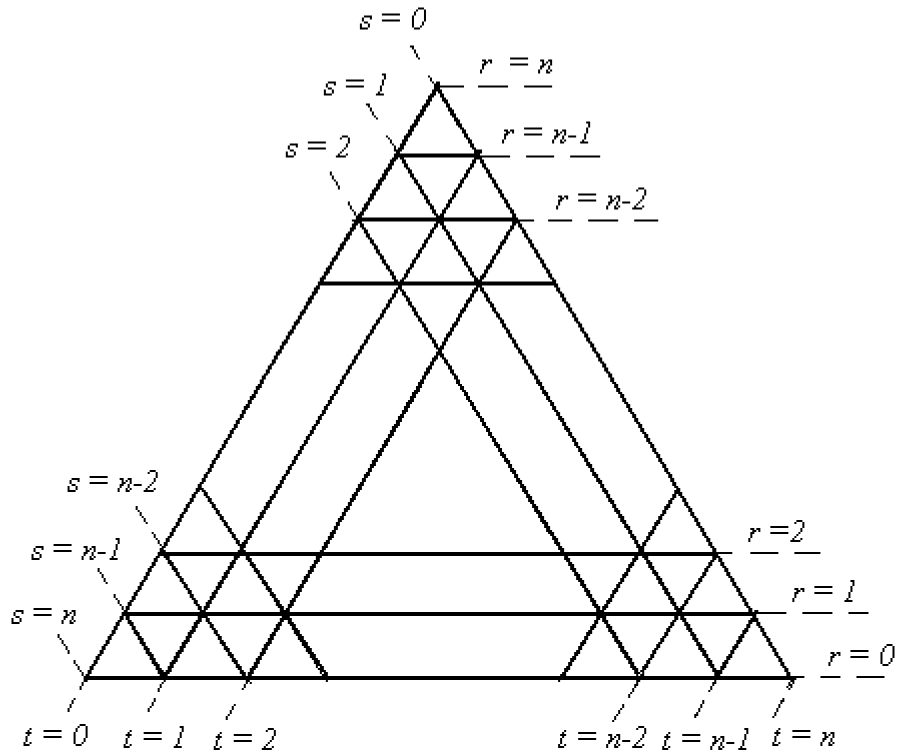

FIGURE 3.1. The triangle with integer barycentric coordinates

Proposition 3.3.1. *All Herrmann's endomorphisms γ_{ij} commute:*

$$\gamma_{1i}\gamma_{1j} = \gamma_{1j}\gamma_{1i}, \quad i, j \in \{2, 3, 4\}, \quad i \neq j. \tag{3.13}$$

Proof. It suffices to check the commutativity on generators. We will check that

$$\gamma_{13}(\gamma_{12}(e_i)) = \gamma_{12}(\gamma_{13}(e_i)), \qquad \text{where } i = 1, 2, 3, 4. \tag{3.14}$$

We have $\gamma_{12}(e_1) = e_1(e_3 + e_4)$, and

$$\begin{aligned}\gamma_{13}(\gamma_{12}(e_1)) =&\gamma_{13}(e_1(e_3 + e_4)) = \gamma_{13}(e_1)(\gamma_{13}(e_3) + \gamma_{13}(e_4)) = \\ & e_1(e_2 + e_4)(e_3(e_2 + e_4) + e_4(e_1 + e_3)) = \\ & e_1(e_2 + e_4)(e_3 + e_4)(e_1 + e_3) = \\ & e_1(e_2 + e_4)(e_3 + e_4),\end{aligned}$$

and

$$\gamma_{13}(\gamma_{12}(e_1)) = \gamma_{12}(\gamma_{13}(e_1)). \tag{3.15}$$

From (3.15) we have

$$\begin{aligned}&\gamma_{13}(\gamma_{12}(e_2)) = \gamma_{24}(\gamma_{21}(e_2)) = \gamma_{21}(\gamma_{24}(e_2)) = \gamma_{12}(\gamma_{13}(e_2)), \\ &\gamma_{13}(\gamma_{12}(e_3)) = \gamma_{31}(\gamma_{34}(e_3)) = \gamma_{34}(\gamma_{31}(e_3)) = \gamma_{12}(\gamma_{13}(e_3)), \\ &\gamma_{13}(\gamma_{12}(e_4)) = \gamma_{42}(\gamma_{43}(e_4)) = \gamma_{43}(\gamma_{42}(e_4)) = \gamma_{12}(\gamma_{13}(e_4)). \quad \square\end{aligned}$$

3.3.1. *More relations on the admissible sequences.* Further, we want to found a connection between Herrmann's endomorphisms γ_{1i} and admissible sequences in D^4. As we will see in Proposition 3.3.4, endomorphisms γ_{1i} add corresponding indices to the start of the given admissible sequence, while elementary maps ϕ_i of Gelfand-Ponomarev add indices to the end of the corresponding admissible sequence, see Table 2.1. To found this connection, we

need more relations connecting admissible sequences in D^4, see Proposition 2.1.1 and the fundamental property of the admissible sequences (2.1).

Proposition 3.3.2. *The following relations hold*
1) $2(13)^s 1 = 2(42)^s 1$,
2) $2(13)^s(14)^r 1 = 2(42)^s(32)^r 1$,
3) $1(41)^r(31)^s(21)^t = (14)^r(13)^s(12)^t 1 = (12)^t(13)^s(14)^r 1$,
4) $1(41)^r(31)^s(21)^t = (12)^t(42)^s(32)^r 1$,
5) $3(41)^s 2 = (32)^s 12$,
6) $3(41)^s(21)^t(31)^r = 3(41)^s(31)^t(21)^r = (32)^s(42)^r(12)^t 1$,
7) $2(42)^r(32)^s(12)^t 1 = (21)^{t+1}(41)^s(31)^r$,
8) $3(42)^s(12)^t(32)^r 1 = (31)^s(41)^r(21)^{t+1}$,
9) $1(42)^r(32)^s(12)^t 1 = (13)^r(41)^{s+1}(21)^t$,
10) $(23)^s(42)^r(12)^t 1 = 2(41)^s(31)^{r-1}(21)^{t+1}$.

Proof. 1) We have

$$\begin{aligned} 2(13)^s 1 =& 2(13)(13)(13)\dots(13)(13)1 = 2(42)(13)(13)\dots(13)(13)1 = \\ & 2(42)(42)(13)\dots(13)(13)1 = 2(42)(42)(42)\dots(42)(42)1 = 2(42)^s 1. \end{aligned}$$

2) By heading 1) we have

$$\begin{aligned} 2(13)^s(14)^r 1 =& 2(42)^s(32)^r 1 = 2(42)^s(14)(14)(14)\dots(14)(14)1 = \\ & 2(42)^s(32)(14)(14)\dots(14)(14)1 = 2(42)^s(32)(32)(32)\dots(32)(32)1 = \\ & 2(42)^s(32)^r 1. \end{aligned}$$

3) First,

$$\begin{aligned} 1(41)^r =& 1(41)(41)\dots(41) = (14)(14)\dots(14)1 = (14)^r 1, \text{ and} \\ & 1(41)^r(31)^s(21)^t = (14)^r(13)^s(12)^t 1. \end{aligned}$$

By heading 6) of Proposition 2.1

$$\begin{aligned} & 1(41)^r(31)^s(21)^t = 1(21)^t(31)^s(41)^r, \text{ and} \\ & 1(41)^r(31)^s(21)^t = (12)^t(13)^s(14)^r 1. \end{aligned}$$

4) By heading 3) and 2) we have

$$\begin{aligned} 1(41)^r(31)^s(21)^t =& (12)^t(13)^s(14)^r 1 = (12)^{t-1}1[2(13)^s(14)^r 1] = \\ & (12)^{t-1}1[2(42)^s(32)^r 1] = (12)^{t-1}12(42)^s(32)^r 1 = \\ & (12)^t(42)^s(32)^r 1. \end{aligned}$$

5) Here,

$$\begin{aligned} 3(41)^s 2 =& 3(41)(41)\dots(41)(41)2 = 3(23)(41)\dots(41)(41)2 = \\ & 3(23)(23)\dots(23)212 = (32)^s 12. \end{aligned}$$

6) By heading 5) of this proposition and by heading 6) of Proposition 2.1we have

$$\begin{aligned} 3(41)^s(21)^t(31)^r =& (32)^s 1(21)^t(31)^r = \\ & (32)^s 1(31)^r(21)^t = (32)^s(13)^r(12)^t 1 = \\ & (32)^s(13)(13)\dots(13)(13)(12)^t 1 = \\ & (32)^s(42)(13)\dots(13)(13)(12)^t 1 = \\ & (32)^s(42)^r(12)^t 1. \end{aligned}$$

7) By heading 2)

$$\begin{aligned} 2(32)^s(42)^r1 = 2(14)^s(13)^r1 = \\ 2(13)^r(14)^s1 = 21(31)^r(41)^s. \end{aligned} \tag{3.16}$$

By (3.16) and by heading 6) of Proposition 2.1 we have

$$\begin{aligned} & 2(42)^r(32)^s(12)^t1 = 2(12)^t(32)^s(42)^r1 = \\ & 2(12)^t1(31)^r(41)^s = (21)^{t+1}(31)^r(41)^s = \\ & (21)^{t+1}(41)^s(31)^r. \end{aligned}$$

8) By heading 5) and substitution $1 \leftrightarrow 2$

$$3(42)^s1 = (31)^s21. \tag{3.17}$$

By (3.17) and by heading 1) of Proposition 2.1 we have

$$\begin{aligned} 3(42)^s(12)^t(32)^r1 = & (31)^s2(12)^t(32)^r1 = (31)^s(21)^t(23)^r21 = \\ & (31)^s(23)^r(21)^{t+1} = (31)^s(23)(23)\dots(23)(23)(21)^{t+1} = \\ & (31)^s(41)(23)\dots(23)(23)(21)^{t+1} = (31)^s(41)^r(21)^{t+1}. \end{aligned}$$

9) By substitution $3 \to 1 \to 2 \to 3$ in heading 5) we have

$$1(42)^r3 = (13)^r23. \tag{3.18}$$

In addition,

$$\begin{aligned} 3(23)^s21 = & 3(23)(23)\dots(23)(23)21 = 3(23)(23)\dots(23)(23)41 \\ & 3(23)(23)\dots(23)(41)41 = 3(41)(41)\dots(41)(41)41 = 3(41)^{s+1}. \end{aligned} \tag{3.19}$$

From (3.18) and (3.19) we get

$$\begin{aligned} 1(42)^r(32)^s(12)^t1 = & (13)^r2(32)^s(12)^t1 = (13)^r(23)^s2(12)^t1 = \\ & (13)^r(23)^s(21)(21)^t = (13)^r(41)^{s+1}(21)^t. \end{aligned} \tag{3.20}$$

10) First, we have

$$\begin{aligned} (23)^s42 = & 2(32)(32)\dots(32)342 = 2(32)(32)\dots(32)412 = \\ & 2(41)(41)\dots(41)412 = 2(41)^s2, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} 1(24)^{r-1}21 = & 1(24)(24)\dots(24)21 = 1(31)(24)\dots(24)21 = \\ & 1(31)(31)\dots(31)21 = 1(31)^{r-1}21. \end{aligned} \tag{3.22}$$

From (3.21) and (3.22) we get

$$\begin{aligned} (23)^s(42)^r(12)^t1 = & (23)^s4(24)^{r-1}2(12)^t1 = \\ & 2(41)^s(24)^{r-1}2(12)^t1 = 2(41)^s(31)^{r-1}2(12)^t1 = \\ & 2(41)^s(31)^{r-1}(21)^{t+1}. \end{aligned} \tag{3.23}$$

The proposition is proved. $\square$

3.3.2. *How Herrmann's endomorphisms act on the admissible elements.* Before understanding how Herrmann's endomorphisms act on the admissible elements, we should know how these endomorphisms work on the simple admissible elements $e_k a^{ij}$ are *almost atomic* elements[7].

Proposition 3.3.3. *Endomorphisms γ_{ij} act on admissible elements as follows:*

(1) $\gamma_{1j}(e_1 a_r^{kl}) = e_1 a_{r+1}^{kl}$, $\{1, j, k, l\} = \{1, 2, 3, 4\}$,

(2) $\gamma_{1k}(e_1 a_r^{kl}) = \gamma_{jl}(e_1 a_r^{kl}) = e_1 a_1^{jl} a_r^{kl}$, $\{1, j, k, l\} = \{1, 2, 3, 4\}$.

Proof. Without lost of generality, we will show that

$$\gamma_{12}(e_1 a_r^{34}) = e_1 a_{r+1}^{34}, \tag{3.24}$$

and

$$\gamma_{12}(e_1 a_r^{23}) = e_1 a_1^{34} a_r^{23}. \tag{3.25}$$

1) Let us prove (3.24). For r=0, we get

$$\gamma_{12}(e_1) = e_1(e_3 + e_4) = e_1 a_1^{34}. \tag{3.26}$$

For r=1,

$$\begin{aligned}
\gamma_{12}(e_1 a_1^{34}) =& \gamma_{12}(e_1(e_3 + e_4)) = \gamma_{12}(e_1)(\gamma_{12}(e_3) + \gamma_{12}(e_4)) = \\
& e_1(e_3 + e_4)(e_3(e_1 + e_2) + e_4(e_1 + e_2)) = e_1(e_3(e_1 + e_2) + e_4(e_1 + e_2)) = \\
& e_1(e_3 + e_4(e_1 + e_2)) = e_1 a_2^{34}.
\end{aligned}$$

By induction hypothesis and (3.10) we have

$$\begin{aligned}
\gamma_{12}(e_1 a_r^{34}) =& \gamma_{12}(e_1(e_3 + e_4 a_{r-1}^{12})) = \gamma_{12}(e_1)(\gamma_{12}(e_3) + \gamma_{12}(e_4)\gamma_{34}(a_{r-1}^{12})) = \\
& e_1(e_3 + e_4)(e_3(e_1 + e_2) + e_4(e_1 + e_2)a_r^{12}) = \\
& e_1(e_3 + e_4)(e_3 + e_4(e_1 + e_2)a_r^{12}) = \\
& e_1(e_3 + e_4)(e_3 + e_4 a_r^{12}) = e_1 a_{r+1}^{34}.
\end{aligned}$$

2) Now, let us prove (3.25). For r=1,

$$\begin{aligned}
\gamma_{12}(e_1 a_1^{23}) =& \gamma_{12}(e_1(e_2 + e_3)) = \gamma_{12}(e_1)(\gamma_{12}(e_2) + \gamma_{12}(e_3)) = \\
& e_1(e_3 + e_4)(e_2(e_3 + e_4) + e_3(e_1 + e_2)) = \\
& e_1(e_2(e_3 + e_4) + e_3(e_1 + e_2)) = \\
& e_1(e_3 + e_4)(e_2 + e_3)(e_1 + e_2) = e_1(e_3 + e_4)(e_2 + e_3)) = \\
& e_1 a_1^{34} a_1^{23}.
\end{aligned}$$

By induction hypothesis, and again by (3.10) we have

$$\begin{aligned}
\gamma_{12}(e_1 a_r^{23}) =& \gamma_{12}(e_1(e_2 + e_3 a_{r-1}^{41})) = \gamma_{12}(e_1)(\gamma_{12}(e_2) + \gamma_{34}(e_3 a_{r-1}^{41})) = \\
& e_1(e_3 + e_4)(e_2(e_3 + e_4) + e_3 a_1^{12} a_{r-1}^{41}) = \\
& e_1(e_3 + e_4)(e_2 + e_3 a_1^{12} a_{r-1}^{41}) = \\
& e_1 a_1^{34}(e_2 + e_3 a_{r-1}^{41}) = e_1 a_1^{34} a_r^{23}.
\end{aligned}$$

The proposition is proved. □

[7] Recall, that polynomials a_r^{ij} are called *atomic*, see (1.12).

Theorem 3.3.4. *Endomorphism γ_{ik} acts on the admissible element $e_{\alpha k}$ as follows:*

$$\gamma_{ik}(e_{i_r i_{r-1}\dots i_2 k}) = e_{i_r i_{r-1}\dots i_2 k i},$$

or, in other words,

$$\boxed{\gamma_{ik}(e_{\alpha k}) = e_{\alpha k i},} \tag{3.27}$$

where $i, k \in \{1,2,3,4\}$, $i \neq k$, and $\alpha, \alpha k$ are admissible sequences.

Without lost of generality, we will show that

$$\gamma_{12}(e_{\alpha 2}) = e_{\alpha 21}. \tag{3.28}$$

It suffices to prove (3.28) for all cases $\alpha 2$ described by Table 2.3 with the permutation $1 \leftrightarrow 2$:

$$F12, F32, F42, G22, G12, G32, G42, H22.$$

<u>Case $F12$</u>. Here,

$$\begin{array}{l} \alpha 2 = (12)^t(42)^r(32)^s, \quad e_{\alpha 2} = e_1 a^{32}_{2r} a^{42}_{2s} a^{34}_{2t-1}, \\ e_\beta = \gamma_{12}(e_{\alpha 2}) = e_1 a^{32}_{2r} a^{42}_{2s} a^{34}_{2t}, \text{ where} \\ \beta = 1(41)^r(31)^s(21)^t, \quad (\text{case } G11). \end{array} \tag{3.29}$$

Thus, by Proposition 3.3.2, heading 4), we have $\beta = (12)^t(42)^r(32)^s 1 = \alpha 21$.

<u>Case $F32$</u>. We have

$$\begin{array}{l} \alpha 2 = (32)^s(42)^r(12)^t, \quad e_{\alpha 2} = e_3 a^{12}_{2t} a^{42}_{2r} a^{14}_{2s-1}, \\ e_\beta = \gamma_{12}(e_{\alpha 2}) = e_3 a^{12}_{2t+1} a^{42}_{2r} a^{14}_{2s-1}, \text{ where} \\ \beta = 3(41)^s(21)^t(31)^r, \quad (\text{case } G31 \text{ with } r \leftrightarrow s). \end{array} \tag{3.30}$$

By Proposition 3.3.2, heading 6), we have $\beta = (32)^s(42)^r(12)^t 1 = \alpha 21$.

<u>Case $G22$</u>. Here,

$$\begin{array}{l} \alpha 2 = 2(42)^r(32)^s(12)^t, \quad e_{\alpha 2} = e_2 a^{14}_{2s} a^{34}_{2t} a^{31}_{2r}, \\ e_\beta = \gamma_{12}(e_{\alpha 2}) = e_2 a^{14}_{2s} a^{34}_{2t+1} a^{31}_{2r}, \text{ where} \\ \beta = (21)^{t+1}(41)^s(31)^r, \quad (\text{case } F21 \text{ with } r \leftrightarrow s, t \to t+1). \end{array} \tag{3.31}$$

By Proposition 3.3.2, heading 7), we have $\beta = (21)^{t+1}(41)^s(31)^r 1 = \alpha 21$.

<u>Case $G32$</u>. We have

$$\begin{array}{l} \alpha 2 = 3(42)^r(12)^t(32)^s, \quad e_{\alpha 2} = e_3 a^{14}_{2s} a^{12}_{2t+1} a^{24}_{2r-1}, \\ e_\beta = \gamma_{12}(e_{\alpha 2}) = e_3 a^{14}_{2s} a^{12}_{2t+2} a^{24}_{2r-1}, \text{ where} \\ \beta = (31)^r(41)^s(21)^{t+1}, \quad (\text{case } F31 \text{ with } r \leftrightarrow s, t \to t+1). \end{array} \tag{3.32}$$

By Proposition 3.3.2, heading 8), we have $\beta = 3(42)^r(12)^t(32)^s 1 = \alpha 21$.

<u>Case $G12$</u>. We have

$$\begin{array}{l} \alpha 2 = 1(42)^r(32)^s(12)^t, \quad e_{\alpha 2} = e_1 a^{34}_{2t} a^{32}_{2s+1} a^{24}_{2r-1}, \\ e_\beta = \gamma_{12}(e_{\alpha 2}) = e_1 a^{34}_{2t+1} a^{32}_{2s+1} a^{24}_{2r-1}, \text{ where} \\ \beta = (13)^r(41)^{s+1}(21)^t, \quad (\text{case } H11 \text{ with } r \leftrightarrow s, s \to s+1). \end{array} \tag{3.33}$$

By Proposition 3.3.2, heading 9), we have $\beta = 1(42)^r(32)^s(12)^t 1 = \alpha 21$.

Case $H22$. In this case we have

$$\begin{aligned} &\alpha 2 = (23)^s(42)^r(12)^t, \quad e_{\alpha 2} = e_2 a^{13}_{2r-1} a^{14}_{2s-1} a^{34}_{2t+1}, \\ &e_\beta = \gamma_{12}(e_{\alpha 2}) = e_2 a^{13}_{2r-1} a^{14}_{2s-1} a^{34}_{2t+2}, \text{ where} \\ &\beta = 2(41)^s(31)^{r-1}(21)^{t+1}, \quad (\text{case } G21, t \to t+1, r \leftrightarrow s, r \to r-1). \end{aligned} \tag{3.34}$$

By Proposition 3.3.2, heading 10), we have $\beta = (23)^s(42)^r(12)^t 1 = \alpha 21$.

We drop case $F42$ (resp. $G42$) which is similar to $F32$ (resp. $G32$) and is proved just by permutation $3 \leftrightarrow 4$. $\square$

Corollary 3.3.5. *For cumulative elements $e(n)$, the following relation holds:*

$$e_i(n+1) = \gamma_{ij}(e_j(n)) + \gamma_{ik}(e_k(n)) + \gamma_{il}(e_l(n)), \tag{3.35}$$

where $\{i,j,k,l\} = \{1,2,3,4\}$.

Proof. Without lost of generality, we will show that

$$e_1(n+1) = \gamma_{12}(e_2(n)) + \gamma_{13}(e_3(n)) + \gamma_{14}(e_4(n)). \tag{3.36}$$

By Theorem 3.3.4 we have

$$\gamma_{12}(e_2(n)) = \sum_{|\alpha|=n-1} e_{\alpha 21},$$

$$\gamma_{13}(e_3(n)) = \sum_{|\alpha|=n-1} e_{\alpha 31},$$

$$\gamma_{14}(e_4(n)) = \sum_{|\alpha|=n-1} e_{\alpha 41},$$

where α in every sum runs over all admissible elements of the length $n-1$. Then,

$$\gamma_{12}(e_2(n)) + \gamma_{13}(e_3(n)) + \gamma_{14}(e_4(n)) = \sum_{|\beta|=n} e_{\beta 1},$$

where β runs over all admissible elements of the length n. The last sum is $e_1(n+1)$ and relation (3.36) is proved. $\square$

Similarly to relation (3.27) for e_α, it can be proved the relation for admissible elements $f_{\alpha 0}$:

$$\gamma_{ik}(f_{\alpha k0}) = f_{\alpha ki0}. \tag{3.37}$$

For example,

$$f_{20} = e_2(e_1 + e_3 + e_4),$$

and

$$\begin{aligned} \gamma_{12}(f_{20}) =& e_2(e_3+e_4)(e_1(e_3+e_4) + e_3(e_1+e_2) + e_4(e_1+e_2)) = \\ & e_2(e_3+e_4)(e_1 + e_3(e_1+e_2) + e_4) = f_{210}, \end{aligned}$$

see Proposition 2.8.5 and Section 1.6.

3.3.3. *A sequence of Herrmann's endomorphisms.* Now, we will see how admissible elements e_α is obtained by means of sequence of Herrmann's endomorphisms.

Theorem 3.3.6 (Admissible elements and Herrmann's endomorphisms)**.** *Admissible elements e_α and $f_{\alpha 0}$ ending at 1 (from Table 3.1) are obtained by means of Herrmann's endomorphisms as follows:*

$$\boxed{\gamma^t_{12}\gamma^s_{13}\gamma^r_{14}(e_1) = e_1 a^{34}_t a^{24}_s a^{32}_r,} \tag{3.38}$$

and

$$\boxed{\gamma^t_{12}\gamma^s_{13}\gamma^r_{14}(f_{10}) = e_1 a^{34}_t a^{24}_s a^{32}_r (e_1 a^{34}_{t+1} + a^{24}_{s+1} a^{32}_{r-1}), \quad \textit{for } r \geq 1.} \tag{3.39}$$

Similarly, admissible elements e_α and $f_{\alpha 0}$ ending at $i = 2, 3, 4$ are obtained as follows:

$$\begin{aligned}
\gamma^t_{12}\gamma^s_{13}\gamma^r_{14}(e_2) &= e_2 a^{34}_t a^{14}_s a^{13}_r, \\
\gamma^t_{12}\gamma^s_{13}\gamma^r_{14}(e_3) &= e_3 a^{14}_t a^{24}_s a^{12}_r, \\
\gamma^t_{12}\gamma^s_{13}\gamma^r_{14}(e_4) &= e_4 a^{31}_t a^{21}_s a^{23}_r,
\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}
\gamma^t_{12}\gamma^s_{13}\gamma^r_{14}(f_{20}) &= e_2 a^{34}_t a^{14}_s a^{31}_r (e_2 a^{34}_{t+1} + a^{14}_{s+1} a^{31}_{r-1}), \quad \textit{for } r \geq 1. \\
\gamma^t_{12}\gamma^s_{13}\gamma^r_{14}(f_{30}) &= e_3 a^{14}_t a^{24}_s a^{12}_r (e_3 a^{14}_{t+1} + a^{24}_{s+1} a^{12}_{r-1}), \quad \textit{for } r \geq 1. \\
\gamma^t_{12}\gamma^s_{13}\gamma^r_{14}(f_{40}) &= e_4 a^{31}_t a^{21}_s a^{32}_r (e_1 a^{31}_{t+1} + a^{21}_{s+1} a^{32}_{r-1}), \quad \textit{for } r \geq 1.
\end{aligned} \tag{3.41}$$

Proof. 1) Consider the action of Herrmann's endomorphisms on e_α. For $t = 1, s = 0, r = 0$ relation (3.38) follows from (3.26). Assume, relation(3.38) is true for $r+s+t = n$. By (3.24) and (2.5) we have

$$\begin{aligned}
\gamma^{t+1}_{12}\gamma^s_{13}\gamma^r_{14}(e_1) = &\gamma_{12}(e_1 a^{34}_t a^{24}_s a^{32}_r) = \gamma_{12}(e_1 a^{34}_t)\gamma_{12}(e_1 a^{24}_s)\gamma_{12}(e_1 a^{23}_r) = \\
&e_1 a^{34}_{t+1} e_1 a^{43}_1 a^{24}_s e_1 a^{34}_1 a^{23}_r = e_1 a^{34}_{t+1} a^{24}_s a^{23}_r.
\end{aligned}$$

Thus, (3.38) is true for all triples (r, s, t).

2) Now, consider action on $f_{\alpha 0}$. For $t = 1, s = 0, r = 0$, we have

$$\begin{aligned}
\gamma_{12}(f_{10}) = &\gamma_{12}(e_1(e_2 + e_3 + e_4)) = \\
&e_1(e_3 + e_4)(e_2(e_3 + e_4) + e_3(e_1 + e_2) + e_4(e_1 + e_2)) = \\
&e_1 a^{34}_1 (e_2 + e_3(e_1 + e_2) + e_4) = e_1 a^{34}_1 (e_2 + e_1(e_3 + e_2) + e_4) = \\
&e_1 a^{34}_1 (e_1 a^{32}_1 + a^{42}_1).
\end{aligned} \tag{3.42}$$

On the other hand, for the case $F12$, $f_{\alpha 0}$ can be written as follows:

$$\begin{aligned}
f_{\alpha 0} = &e_\alpha(e_1 a^{34}_{2t} + a^{42}_{2s+1} a^{32}_{2r-1}) = \\
&e_\alpha(e_1 a^{34}_{2t} a^{32}_{2r-1} + a^{42}_{2s+1}) = \\
&e_\alpha(e_1 a^{34}_{2t-2} a^{32}_{2r+1} + a^{42}_{2s+1}).
\end{aligned} \tag{3.43}$$

We use the last expression of $f_{\alpha 0}$ in (3.43) for the case $t = 1, r = 0, s = 0$. Then,

$$f_{\alpha 0} = e_\alpha(e_1 a^{34}_0 a^{32}_1 + a^{42}_1) = e_\alpha(e_1 a^{32}_1 + a^{42}_1). \tag{3.44}$$

Thus, by (3.42) and (3.44) relation (3.39) is true for $t = 1, s = 0, r = 0$.

By (3.39) and Proposition 3.3.3 we have the induction step:

$$\begin{aligned}\gamma_{12}^{t+1}\gamma_{13}^{s}\gamma_{14}^{r}(f_{\alpha 0}) =&\gamma_{12}(e_1 a_t^{34} a_s^{24} a_r^{32}(e_1 a_{t+1}^{34} + a_{s+1}^{24} a_{r-1}^{32})) = \\ &\gamma_{12}(e_1 a_t^{34})\gamma_{12}(e_1 a_s^{24})\gamma_{12}(e_1 a_r^{32})(\gamma_{12}(e_1 a_{t+1}^{34}) + \gamma_{12}(e_1 a_{s+1}^{24})\gamma_{12}(a_{r-1}^{32})) = \\ &(e_1 a_{t+1}^{34})(e_1 a_1^{34} a_s^{24})(e_1 a_1^{43} a_r^{23})(e_1 a_{t+2}^{34} + (e_1 a_1^{34} a_{s+1}^{24})(e_1 a_1^{43} a_{r-1}^{23})) = \\ &e_1 a_{t+1}^{34} a_s^{24} a_r^{23}(e_1 a_{t+2}^{34} + e_1 a_1^{34} a_{s+1}^{24} e_1 a_{r-1}^{23}) = \\ &e_1 a_{t+1}^{34} a_s^{24} a_r^{23}(e_1 a_{t+2}^{34} + a_{s+1}^{24} e_1 a_{r-1}^{23}). \quad \square\end{aligned}$$

3.3.4. *The sum of Herrmann's endomorphisms.* Consider endomorphism $\mathcal{R}$ is the sum of Herrmann's endomorphisms γ_{1i}:

$$\mathcal{R} = \gamma_{12} + \gamma_{13} + \gamma_{14}. \tag{3.45}$$

Proposition 3.3.7. *The endomorphism $\mathcal{R}$ relates the inverse cumulative elements (3.5) as follows:*

$$\mathcal{R}e_i^{\vee}(n) = e_i^{\vee}(n+1), \qquad \textit{where } i = 1,2,3,4. \tag{3.46}$$

and

$$\mathcal{R}f_0^{\vee}(n) = f_0^{\vee}(n+1). \tag{3.47}$$

Proof. 1) By (3.5)

$$e_1^{\vee}(n) = \sum_{r+s+t=n-1} e_1 a_r^{32} a_s^{24} a_t^{34},$$

and by Proposition 3.3.3 we have

$$\begin{aligned}\mathcal{R}(e_1^{\vee}(n)) =&(\gamma_{12} + \gamma_{13} + \gamma_{14}) \sum_{r+s+t=n-1} e_1 a_r^{32} a_s^{24} a_t^{34} = \\ &\sum_{r+s+t=n-1} e_1 a_r^{32} a_s^{24} a_{t+1}^{34} + \sum_{r+s+t=n-1} e_1 a_r^{32} a_{s+1}^{24} a_t^{34} + \sum_{r+s+t=n-1} e_1 a_{r+1}^{32} a_s^{24} a_t^{34} = \\ &\sum_{r+s+t=n} e_1 a_r^{32} a_s^{24} a_t^{34} = e_1^{\vee}(n+1).\end{aligned}$$

2) By (3.6)

$$f_0^{\vee}(n) = \sum_{r+s+t=n} e_\alpha (a_r^{jk} + a_s^{kl} a_t^{lj}),$$

and, again, by Proposition 3.3.3

$$\begin{aligned}\mathcal{R}(f_0^{\vee}(n)) =&(\gamma_{12} + \gamma_{13} + \gamma_{14}) \sum_{r+s+t=n} e_\alpha (a_r^{jk} + a_s^{kl} a_t^{lj}) = \\ &(\gamma_{12} + \gamma_{13} + \gamma_{14}) \sum_{r+s+t=n} \gamma_{12}^{t}\gamma_{23}^{s}\gamma_{14}^{r}(f_{i0}), \\ &\text{where } \{i,j,k,l\} = \{1,2,3,4\}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{R}(f_0^{\vee}(n)) =& \sum_{r+s+t=n} \gamma_{12}^{t+1}\gamma_{23}^{s}\gamma_{14}^{r}(f_{i0}) + \\ &\sum_{r+s+t=n} \gamma_{12}^{t}\gamma_{23}^{s+1}\gamma_{14}^{r}(f_{i0}) + \sum_{r+s+t=n} \gamma_{12}^{t}\gamma_{23}^{s}\gamma_{14}^{r+1}(f_{i0}) = \\ &\sum_{r+s+t=n+1} \gamma_{12}^{t}\gamma_{23}^{s}\gamma_{14}^{r}(f_{i0}) = f_0^{\vee}(n+1). \quad \square\end{aligned}$$

3.4. **Perfect elements in D^4.**

3.4.1. *The Gelfand-Ponomarev conjecture.* By Gelfand-Ponomarev definition (1.19), elements $h_i(n)$, where $i = 1,2,3,4$ are generators of the 16-element Boolean cube $B^+(n)$ of perfect elements. According to (1.19), these 16 elements in $B^+(n)$ are as follows:

$$\begin{aligned} h^{\max}(n) &= \sum_{i=1,2,3,4} h_i(n) = \sum_{i=1,2,3,4} e_i(n), \\ h^{\min}(n) &= \bigcap_{i=1,2,3,4} h_i(n) = \sum_{i=1,2,3,4} e_i(n)h_i(n), \end{aligned} \tag{3.48}$$

and

$$\begin{aligned} & h_i(n) = e(j) + e(k) + e(l), \\ & h_i(n)h_j(n) = e_i(n)h_i(n) + e_j(n)h_j(n) + e_k(n) + e_l(n), \\ & h_i(n)h_j(n)h_k(n) = e_i(n)h_i(n) + e_j(n)h_j(n) + e_k(n)h_k(n) + e_l(n), \\ & \qquad\qquad \text{where } \{i,j,k,l\} = \{1,2,3,4\}, \end{aligned} \tag{3.49}$$

see also Table 1.1.

Four elements $h_i(n)$ (resp. four elements $h_i(n)h_j(n)h_k(n)$) from (3.49) are coatoms (resp. atoms) in $B^+(n)$. The maximal and minimal elements in $B^+(1)$ are as follows:

$$\begin{aligned} & h^{\max}(1) = e_1 + e_2 + e_3 + e_4, \\ & h^{\min}(1) = (e_1 + e_2 + e_3)(e_1 + e_2 + e_4)(e_1 + e_3 + e_4)(e_2 + e_3 + e_4). \end{aligned} \tag{3.50}$$

By (3.50) and Section 1.6 we have

$$\begin{aligned} h^{\min}(1) = & e_1(e_2 + e_3 + e_4) + e_2(e_1 + e_3 + e_4) + e_3(e_1 + e_2 + e_4) = \\ & f_{10} + f_{20} + f_{30}. \end{aligned} \tag{3.51}$$

By (3.50) also

$$h^{\min}(1) \supset e_4(e_1 + e_2 + e_3) = f_{40},$$

and by (3.51) we obtain

$$h^{\min}(1) = f_{10} + f_{20} + f_{30} + f_{40},$$

Thus,

$$h^{\min}(1) = f_0(2). \tag{3.52}$$

Relation (3.52) takes place modulo linear equivalence for any n:

$$h^{\min}(n) \simeq f_0(n+1). \tag{3.53}$$

Relation (3.53) is proved like the similar relation for $D^{2,2,2}$, see [St04, Prop. 3.4.2]. According to the Gelfand-Ponomarev conjecture [GP74, p.7], the relation (3.53) takes place:

$$h^{\min}(n) = f_0(n+1) \quad \text{(Gelfand-Ponomarev conjecture)}. \tag{3.54}$$

Further, by (3.49)

$$\begin{aligned} h_1(1)h_2(1)h_3(1) = & (e_2 + e_3 + e_4)(e_1 + e_3 + e_4)(e_1 + e_2 + e_4) = \\ & e_1(e_2 + e_3 + e_4) + e_2(e_1 + e_3 + e_4) + e_3(e_1 + e_2 + e_4) + e_4 = \\ & f_{10} + f_{20} + f_{30} + e_4 = e_4 + f_0(2). \end{aligned} \tag{3.55}$$

Proposition 3.4.1. *Elements $h_i(n)h_j(n)h_k(n)$ and $e_l(n) + f_0(n+1)$ coincide modulo linear equivalence for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$:*

$$h_i(n)h_j(n)h_k(n) \simeq e_l(n) + f_0(n+1). \tag{3.56}$$

Proof. Assuming that (3.56) is true, we will prove it for $n+1$:

$$h_i(n+1)h_j(n+1)h_k(n+1) \simeq e_l(n+1) + f_0(n+2). \tag{3.57}$$

Similarly to [St04, Prop. 3.4.1] for $D^{2,2,2}$, we have the following relation in D^4 for sums and intersections, commuting on the perfect elements v_t:

$$\sum_{p\,=\,1,2,3,4} \varphi_p \rho_{X^+}(\bigcap_{t\,=\,i,j,k} v_t) = \bigcap_{t\,=\,i,j,k} (\sum_{p\,=\,1,2,3,4} \varphi_p \rho_{X^+}(v_t)), \tag{3.58}$$

where all v_t are perfect elements[8]. Since

$$\sum_{p\,=\,1,2,3,4} \varphi_p \rho_{X^+}(h_t(n)) = \rho_X(h_t(n+1)),$$

we have in the right side of (3.58), that

$$\bigcap_{t\,=\,i,j,k} (\sum_{p\,=\,1,2,3,4} \varphi_p \rho_{X^+}(v_t)) = \rho_X(\bigcap_{t=i,j,k} h_t(n+1)). \tag{3.59}$$

In the left side of (3.58) we have

$$\begin{aligned}\sum_{p\,=\,1,2,3,4} \varphi_p \rho_{X^+}(\bigcap_{t\,=\,i,j,k} h_t(n)) = \sum_{p\,=\,1,2,3,4} \varphi_p \rho_{X^+}(e_l(n) + f_0(n+1)) = \\ \rho_X(e_l(n+1) + f_0(n+2)).\end{aligned} \tag{3.60}$$

Relation (3.57) follows from (3.59) and (3.60). □

Similarly, other relations from Table 1.1 take place modulo linear equivalence:

$$\begin{aligned}&h_i(n)h_j(n)h_k(n) \simeq e_l(n) + f_0(n+1),\\ &h_i(n)h_j(n) \simeq e_k(n) + e_l(n) + f_0(n+1),\\ &h_i(n) \simeq e_j(n) + e_k(n) + e_l(n) + f_0(n+1).\end{aligned} \tag{3.61}$$

Proposition 3.4.2. *The following relations hold:*

$$\begin{aligned}&h_i(n)h_j(n)h_k(n) + h_i(n)h_j(n)h_l(n) = h_i(n)h_j(n),\\ &h_i(n)h_j(n) + h_i(n)h_k(n) = h_i(n).\end{aligned} \tag{3.62}$$

Proof. It easily follows from (3.49) and (1.19). □

---

[8]In the proof of Proposition 3.4.1. from [St04] we only change $\sum \varphi_i \rho_{X^+}(I)$. According to Corollary 2.3.2, heading (3), in the case of D^4 we have

$$\sum_{i=1,2,3,4} \varphi_i \rho_{X^+}(I) = \sum_{i=1,2,3,4} \rho_X(e_i(e_j + e_k + e_l)) = \rho_X(\bigcap_{i=1,2,3,4} (e_j + e_k + e_l)).$$

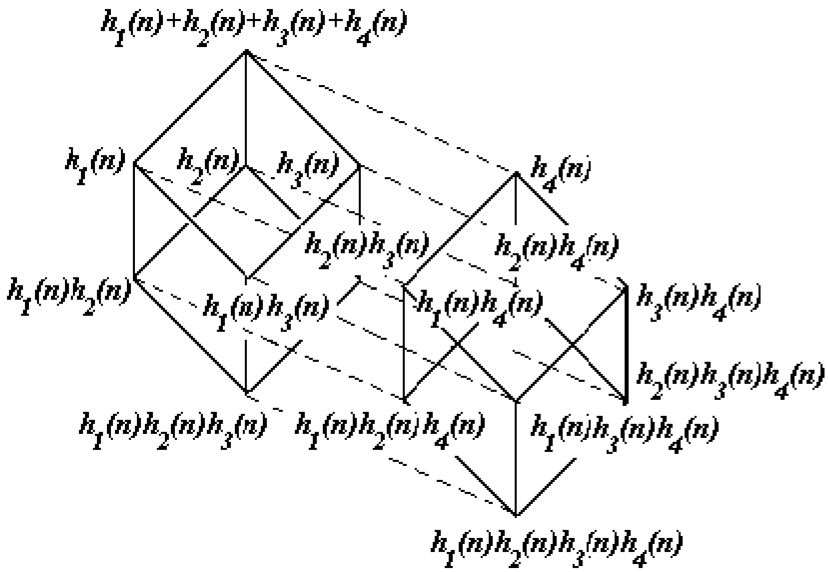

FIGURE 3.2. The 16-element Boolean cube $B^+(n)$ with generators $h_i(n)$

3.4.2. *Herrmann's polynomials s_n, t_n and $p_{i,n}$.* C. Herrmann used endomorphisms γ_{ij} from (3.12) for the construction of perfect elements s_n, t_n and $p_{i,n}$, where $i = 1,2,3,4$, see [H82, p. 362], [H84, p. 229]. In what follows, the definitions of

Polynomials s_n:

$$
\begin{aligned}
& s_0 = I, \quad s_1 = e_1 + e_2 + e_3 + e_4, \\
& s_n^{i-1} = \gamma_{1i}(s_n), \ \text{where } i = 2,3,4, \\
& s_{n+1} = s_n^1 + s_n^2 + s_n^3.
\end{aligned} \tag{3.63}
$$

Polynomials t_n:

$$
\begin{aligned}
& t_0 = I, \quad t_1 = (e_1 + e_2 + e_3)(e_1 + e_2 + e_4)(e_1 + e_3 + e_4)(e_2 + e_3 + e_4), \\
& t_n^{i-1} = \gamma_{1i}(t_n), \ \text{where } i = 2,3,4, \\
& t_{n+1} = t_n^1 + t_n^2 + t_n^3.
\end{aligned} \tag{3.64}
$$

Polynomials $p_{k,n}$:

$$
\begin{aligned}
& p_{i,0} = I, \quad p_{i,1} = e_i + t_1, \ \text{where } i = 1,2,3,4, \\
& p_{i,n+1} = \gamma_{ij}(p_{j,n}) + \gamma_{ik}(p_{k,n}) + \gamma_{il}(p_{l,n}), \\
& \qquad \text{where } i = 1,2,3,4, \ \text{and } \{i,j,k,l\} = \{1,2,3,4\}.
\end{aligned} \tag{3.65}
$$

For example, by (3.45) we have

$$
\begin{aligned}
p_{1,2} = & \gamma_{12}(p_{2,1}) + \gamma_{13}(p_{3,1}) + \gamma_{14}(p_{4,1}) = \gamma_{12}(e_2) + \gamma_{13}(e_3) + \gamma_{14}(e_4) + \mathcal{R}(t_1) = \\
& e_2(e_3 + e_4) + e_3(e_2 + e_4) + e_4(e_2 + e_3) + t_2 = e_{21} + e_{31} + e_{41} + t_2 = \\
& e_1(2) + t_2.
\end{aligned} \tag{3.66}
$$

Similarly,

$$
p_{i,2} = e_i(2) + t_2.
$$

By definitions (3.63), (3.64) we have

$$\begin{aligned} s_{n+1} &= \mathcal{R}s_n = \mathcal{R}^n s_1, \\ t_{n+1} &= \mathcal{R}t_n = \mathcal{R}^n t_1, \end{aligned} \tag{3.67}$$

see Fig. 3.3.

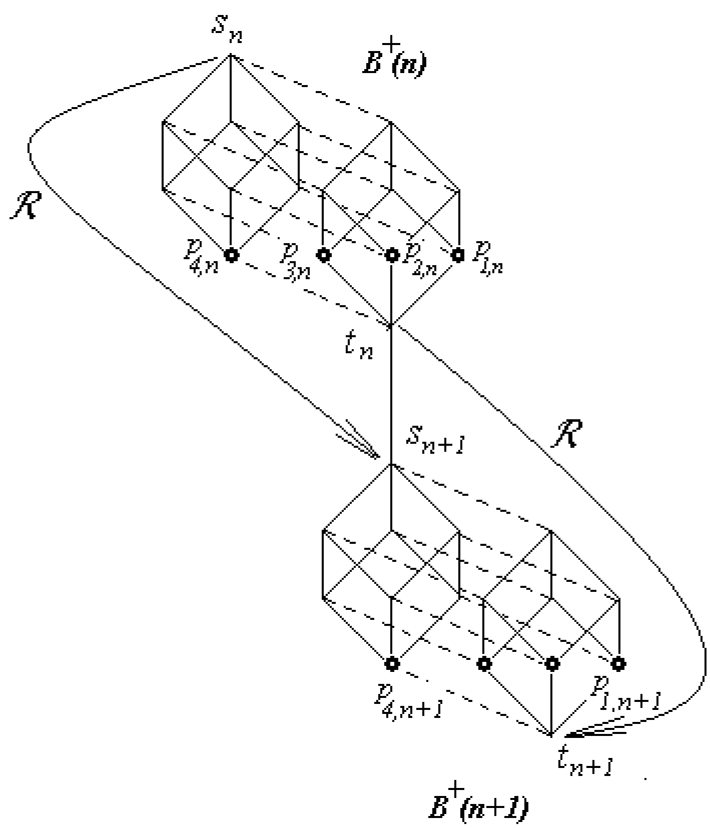

FIGURE 3.3. The endomorphism $\mathcal{R}$ mapping s_n to s_{n+1} and t_n to t_{n+1}

3.4.3. *Cumulative elements and Herrmann's polynomials.* In what follows, we show how Herrmann's polynomials s_n, t_n and $p_{i,n}$, where $i \in \{1,2,3,4\}$, are calculated by means of cumulative elements $e(n)$, $f(n)$. Polynomials s_n and t_n can be also calculated by means of inverse cumulative elements $e^\vee(n)$, $f^\vee(n)$, see Section 3.2. As above, the main elementary brick in these constructions is an admissible element, obtained as a sequence of Herrmann's endomorphisms

$$\gamma_{12}^r \gamma_{13}^s \gamma_{14}^t (e_i),$$

see Theorem 3.3.6 from Section 3.3.3.

Theorem 3.4.3 (The polynomials s_n, t_n and $p_{i,n}$)**.** *1) For each n, the polynomial s_n is the maximal perfect element in the Boolean cube $B^+(n)$, see Section 1.8.2, and it is expressed*

as follows:

$$s_n = \mathcal{R}^{n-1}(s_1) = \sum_{r+s+t=n-1} \gamma_{12}^t \gamma_{13}^s \gamma_{14}^r (e_1 + e_2 + e_3 + e_4) =$$

$$\sum_{i=1,2,3,4} e_i^\vee(n) = \sum_{i=1,2,3,4} e_i(n). \tag{3.68}$$

2) For each n, the polynomial t_n is linearly equivalent to the minimal perfect element in the Boolean cube $B^+(n)$, see Section 1.8.2, and it is expressed as follows:

$$t_n = \mathcal{R}^{n-1}(t_1) =$$

$$\sum_{r+s+t=n} \gamma_{12}^t \gamma_{13}^s \gamma_{14}^r (e_1(e_2 + e_3 + e_4) + e_2(e_1 + e_3 + e_4) + e_3(e_1 + e_2 + e_4)) = \tag{3.69}$$

$$f_0^\vee(n+1) = f_0(n+1).$$

3) For each n, the polynomial $p_{i,n}$ is linearly equivalent to the element $h_j(n)h_k(n)h_l(n)$ in the Boolean cube $B^+(n)$, see Table 1.1, and it is expressed as follows:

$$p_{i,n} = e_i(n) + f_0^\vee(n+1) = e_i(n) + f_0(n+1). \tag{3.70}$$

Proof. 1) By definition (3.63) of s_n, and by property (3.24), for $n = 1$, we have

$$s_2 = \gamma_{12}(s_1) + \gamma_{13}(s_1) + \gamma_{14}(s_1) = \sum_{i=1,2,3,4} (\gamma_{12} + \gamma_{13} + \gamma_{14})(e_i) =$$

$$\sum_{\substack{j=1,2,3 \\ i=1,2,3,4}} \gamma_{1j}(e_i) = \sum_{\substack{i=1,2,3,4 \\ k,l\neq i}} e_i a_1^{kl} = \sum_{|\alpha|=2} e_\alpha = e_1^\vee(2) + e_2^\vee(2) + e_3^\vee(2) + e_4^\vee(2). \tag{3.71}$$

In (3.71) $|\alpha|$ is the length of sequence α. Note, that in (3.71) $|\alpha| = 2$ and $r + s + t = 1$. Assume,

$$s_n = \sum_{|\alpha|=n} e_\alpha = e_1^\vee(n) + e_2^\vee(n) + e_3^\vee(n) + e_4^\vee(n), \tag{3.72}$$

Then, by definition (3.63) by Proposition 3.3.7 and Proposition 3.2.1 we have

$$s_{n+1} = (\gamma_{12} + \gamma_{13} + \gamma_{14}) \sum_{|\alpha|=n} e_\alpha = (\gamma_{12} + \gamma_{13} + \gamma_{14}) \sum_{\substack{i=1,2,3,4 \\ r+s+t=n-1}} \gamma_{12}^r \gamma_{13}^s \gamma_{14}^t (e_i) =$$

$$\sum_{\substack{i=1,2,3,4 \\ r+s+t=n}} \gamma_{12}^r \gamma_{13}^s \gamma_{14}^t (e_i) = \sum_{|\alpha|=n+1} e_\alpha = \sum_{i=1,2,3,4} e_i^\vee(n+1) = \sum_{i=1,2,3,4} e_i(n+1). \tag{3.73}$$

and s_{n+1} is the maximal element in $B^+(n+1)$, see Section 1.8.2.

2) It is easy to see that t_1 in (3.64) is as follows:

$$\begin{aligned} t_1 =& e_1(e_2+e_3+e_4)+e_2(e_1+e_3+e_4)+e_3(e_1+e_2+e_4) = \\ & f_{10}+f_{20}+f_{30} = f_0(2), \end{aligned} \tag{3.74}$$

and by (3.39) from Theorem 3.3.6 we have

$$\begin{aligned} t_2 =& \mathcal{R}(f_0(2)) = (\gamma_{12}+\gamma_{13}+\gamma_{14}) \sum_{i=1,2,3,4} f_{i0} = \sum_{i\neq j} f_{ij0} = \\ & \sum_{|\alpha|=2} f_{\alpha 0} = f_0(3). \end{aligned} \tag{3.75}$$

Assume, (3.69) is true for some n, then by (3.5) we have

$$t_n = \sum_{\substack{i=1,2,3,4 \\ r+s+t=n}} \gamma_{12}^r \gamma_{13}^s \gamma_{14}^t (f_{i0}) = f_0^{\vee}(n+1). \tag{3.76}$$

Then, by Proposition 3.3.7 and Theorem 3.3.6 we have

$$t_{n+1} = Rf_0^{\vee}(n) = f_0^{\vee}(n+1) = \sum_{\substack{i=1,2,3,4 \\ r+s+t=n+1}} \gamma_{12}^r \gamma_{13}^s \gamma_{14}^t (f_{i0}). \tag{3.77}$$

By Proposition 3.2.1 $f_0^{\vee}(n) = f(n)$ and according to (3.53), the element $t_{n+1} = f_0(n+1)$ is linearly equivalent to the minimal element in $B^+(n+1)$. If the Gelfand-Ponomarev conjecture (3.54) is true, the element $t_{n+1} = f_0(n+1)$ coincides with the minimal element $h^{\min}(n+1)$ of $B^+(n+1)$, see Section 3.4.1.

3) For n = 1, $p_{i,1} = e_i + t_1 = e_i(1) + f_0(2)$, and

$$\begin{aligned} p_{i,2} =& \gamma_{ji}(e_j+t_1) + \gamma_{ki}(e_k+t_1) + \gamma_{li}(e_l+t_1) = \\ & e_i(2) + \mathcal{R}t_1 = e_i(2) + t_2 = e_i(2) + f_0(3), \end{aligned} \tag{3.78}$$

see (3.66), (3.75). Further, by Corollary 3.3.5 we have

$$\begin{aligned} p_{i,n+1} =& \gamma_{ji}(p_{j,n}) + \gamma_{ki}(p_{k,n}) + \gamma_{li}(p_{l,n}) = \\ & \gamma_{ji}(e_j(n)+t_n) + \gamma_{ki}(e_k(n)+t_n) + \gamma_{li}(e_l(n)+t_n) = \\ & \gamma_{ji}(e_j(n)) + \gamma_{ki}(e_k(n)) + \gamma_{li}(e_l(n)) + R(t_n) = e_i(n+1) + t_{n+1}. \end{aligned} \tag{3.79}$$

By heading 2)

$$p_{i,n+1} = e_i(n+1) + t_{n+1} = e_i(n+1) + f_0(n+1). \tag{3.80}$$

The theorem is proved. □

As corollary we obtain the following

Theorem 3.4.4 (C. Herrmann, [H82])**.** *Polynomials $p_{1,n}, p_{2,n}, p_{3,n}, p_{4,n}$ are atoms in* 16-*element Boolean cube $B^+(n)$ of perfect elements and*

$$p_{i,n} \simeq h_j(n)h_k(n)h_l(n), \quad \textit{where } \{i,j,k,l\} = \{1,2,3,4\}, \tag{3.81}$$

Polynomials s_n and t_n are, respectively, the maximal and minimal elements in $B^+(n)$, and

$$s_n \simeq \sum_{i=1,2,3,4} h_j(n), \quad t_n \simeq \bigcap_{1=1,2,3,4} h_j(n), \tag{3.82}$$

see Table 1.1.

Proof. From (3.70) and (3.61) we obtain (3.81). From (1.23), (3.69) and (3.48) we obtain (3.82). □

References

[BGP73] Bernstein J. N., Gelfand I. M., Ponomarev V. A., *Coxeter functors and Gabriel's theorem.* (Russian) Uspehi Mat. Nauk 28 (1973), no. 2(170), 19–33. English translation: Russian Math. Surveys, 28(1973), 17–32.

[Bm96] Bogomolny A., *Barycentric coordinates, A Curious Application*, http://www.cut-the-knot.org/triangle/glasses.shtml , *Barycentric Coordinates: A Tool*, http://www.cut-the-knot.org/Curriculum/Geometry/Barycentric.shtml, 1996-2006.

[Cox89] Coxeter H. S. M. *Introduction to Geometry*, 2nd ed. New York, John Wiley & Sons, 1989, 469 pp.

[Cyl93] Cylke A. A., *Perfect and linearly equivalent elements of modular lattices.* CMS Conference Proceedings, (1993), vol. 14, 125–148.

[DR80] Dlab V., Ringel C. M., *Perfect elements in the free modular lattices.* Math. Ann. 247 (1980), 95–100.

[GP70] Gelfand I. M., Ponomarev V. A., *Problems of linear algebra and classification of quadruples of subspaces in a finite-dimensional vector space.* Hilbert space operators and operator algebras (Proc. Internat. Conf., Tihany, 1970), pp. 163–237. Colloq. Math. Soc. Janos Bolyai, 5, North-Holland, Amsterdam, 1972.

[GP74] Gelfand I. M., Ponomarev V. A., *Free modular lattices and their representations.* (Russian) Collection of articles dedicated to the memory of Ivan Georgievic Petrovskii (1901-1973), IV. Uspehi Mat. Nauk 29 (1974), no. 6(180), 3–58. English translation: Russian Math. Surveys 29 (1974), no. 6, 1–56.

[GP76] Gelfand I. M., Ponomarev V. A., *Lattices, representations, and their related algebras.* I. (Russian) Uspehi Mat. Nauk 31 (1976), no. 5(191), 71–88. English translation: Russian Math. Surveys 31 (1976), no. 5, 67–85.

[GP77] Gelfand I. M., Ponomarev V. A., *Lattices, representations, and their related algebras.* II. (Russian) Uspehi Mat. Nauk 32 (1977), no. 1(193), 85–106. English translation: Russian Math. Surveys 32 (1977), no. 1, 91–114.

[HHu75] Herrmann C., Huhn A. P., *Zum Wortproblem für freie Untermodulverbände.* Arch. Math. (Basel) 26 (1975), no. 5, 449–453.

[H82] Herrmann C., *Rahmen und erzeugende Quadrupel in modularen Verb̈nden.* (German) [Frames and generating quadruples in modular lattices] Algebra Universalis 14 (1982), no. 3, 357–387.

[H84] Herrmann C. *On elementary Arguesian lattices with four generators.* Algebra Universalis 18 (1984), no. 2, 225–259.

[Naz67] Nazarova L. A. *Representations of a tetrad.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967) 1361–1378.

[Naz73] Nazarova L. A., *Representations of quivers of infinite type.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat., (1973), 37, N 4. p. 752–791. English translation: Math. USSR-Izv. 7 (1973), 749–792.

[NR72] Nazarova L. A., Roiter A. V., *Representations of partially ordered sets.* (Russian) Investigations on the theory of representations. Zap. Nauchn. Semin, Leningr. Otd. Mat. Inst. Steklova, 1972, 28, p. 5-31. English translation: J. Sov. Math., 1975., 3., p. 585–606.

[Pon91] Ponomarev V. A., *Lattices $L(A_n^\nu)$. Representations of finite-dimensional algebras.* (Tsukuba, 1990), 249–278, CMS Conf. Proc., 11, Amer. Math. Soc., Providence, RI, 1991.

[R97] Rota G-C., *The many lives of lattice theory.* Notices Amer. Math. Soc. 44 (11) (1997), 1440–1445.

[R98] Rota G-C., *Ten mathematics problems I will never solve.* Mitt. Dtsch. Math.-Ver. 1998, no. 2, 45–52.

[St84] Stekolshchik R. B., *Invariant elements in modular lattice.* (Russian) Functional analysis and its applications, 18 (1984) no.1, 82–83. English translation: Funct.Anal. Appl. 18 (1984), no. 1, 73–75.

[St89] Stekolshchik R. B., *Perfect elements in modular lattice, associated with extended Dynkin diagram* $\widetilde{\mathbb{E}}_6$. (Russian) Functional analysis and its applications, 23 (1989) no.3, 90–92. English translation: Funct.Anal. Appl. 3(1989), p.251–254.

[St92] Stekolshchik R. B., *Invariant elements in modular lattice, associated with extended Dynkin diagram* $\widetilde{\mathbb{E}}_6$. (Russian) Functional analysis and its applications, 26 (1992) no.1, 42–53. English translation: Funct.Anal. Appl.(1992), p.33–42.

[St04] Stekolshchik R. B., *Admissible and Perfect Elements in the Modular Lattice*, preprint, 152p., http://arxiv.org/abs/math/0404171, v1, 2004; revised v3, 2005.